

第二部分 数学物理方程

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教材: 林琮桂 <数学物理方法>

Chap. 6 数学物理方程的导出

引言 & 综述 (PPT)

§6.1 简介

含有未知函数及其偏导数的方程称为偏微分方程 (partial differential equation). 如果方程只包含未知函数及其偏导数的一次项, 则称为线性偏微分方程. 方程中出现的偏导数的最高阶数称为方程的阶. 本课程只研究二阶线性偏微分方程. 以两个自变量的情况为例, 记自变量为 x, y , 未知函数为 $u(x, y)$, 则二阶线性偏微分方程的一般形式为

$$a_{11} \frac{\partial^2 u}{\partial x^2} + 2a_{12} \frac{\partial^2 u}{\partial x \partial y} + a_{22} \frac{\partial^2 u}{\partial y^2} + b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} + cu + f = 0, \quad (1)$$

↓
 x, y

↓
 x, y 的已知函数
与 u 无关

$f = 0$ 齐次方程

$f \neq 0$ 非齐次方程 email:

emergence 演生

至多二次导数

二次曲线 $a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + b_1x + b_2y + c = 0$

→ 标准形式 {
双曲线
抛物线
椭圆

§6.1.1 波动方程

在三维空间，波动方程的标准形式为

$$\frac{\partial^2 u}{\partial t^2} - a^2 \nabla^2 u = f, \quad (2a)$$

$$\frac{\partial^2 u}{\partial t^2} \equiv \partial_t^2 u \equiv u_{tt}$$

$$\nabla^2 \equiv \Delta \quad \Delta_3$$

其中 $u = u(\mathbf{r}, t)$ 是我们所要研究的物理量，比如位移或电场 (磁场) 的一个分量， $\mathbf{r}$ 是空间位置， t 是时间， a 是常数，表示波动的传播速度， $f = f(\mathbf{r}, t)$ 是已知函数，代表波动的源， ∇^2 是 Laplace 算符，有些书上记作 Δ ，在直角坐标系中，其他坐标系？

如外力

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

但读者应当注意，这并不是 Laplace 算符的定义。Laplace 算符的定义是 $\nabla^2 u = \nabla \cdot \nabla u$ ，这具有确定的几何意义，而不依赖于坐标系。在一维空间，波动方程的标准形式为

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f, \quad u_{tt} - a^2 u_{xx} = 0 \quad (2b)$$

这可以描写弦的横振动或弹性细杆的纵振动。在数学上，波动方程属于双曲型。

§6.1.2 热传导方程和扩散方程

热传导方程和扩散方程统称为输运方程。在三维空间，它们的标准形式为

$$\frac{\partial u}{\partial t} - a^2 \nabla^2 u = f, \quad (3)$$

其中 $u = u(\mathbf{r}, t)$ 是我们所要研究的物理量，对于热传导方程，它表示温度的分布，对于扩散方程，它表示杂质浓度的分布。 $f = f(\mathbf{r}, t)$ 是已知函数，代表热源或杂质源。在数学上，输运方程属于抛物型。

u_t 与 u_{tt} 差别： $t \rightarrow -t$

对称性 墨水

§6.1.3 稳定场方程

稳定场方程的标准形式为

$$\nabla^2 u = f, \quad (4)$$

其中 $u = u(\mathbf{r})$ 可以代表稳定温度分布、稳定浓度分布或静电场的电势，相应地， $f = f(\mathbf{r})$ 代表不随时间变化的热源、杂质源或自由电荷分布。上式又称为 Poisson 方程，当 $f = 0$ 时又称为 Laplace 方程。在数学上，稳定场方程属于椭圆型。

§6.2 *二阶线性偏微分方程的分类与举例

双曲型

抛物型

椭圆型

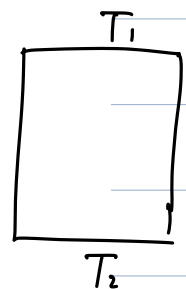
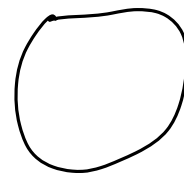
数学物理方程

物理规律: PDE

- ▶ 物理规律的普遍性: 数学物理方程
- ▶ 具体问题的特殊性: 边界条件、初始条件 (统称定解条件)

研究问题的步骤:

- (1) 写出定解问题
- (2) 求解
- (3) 分析解的物理意义和适定性



数学物理方程的导出:

- (1) 确定要研究哪一个物理量 u
- (2) 从所研究的系统中划出一小部分，根据物理规律分析邻近部分和这个小部分的相互作用
- (3) 该相互作用在一个短时间段里怎样影响物理量 u ，得到数学物理方程

§6.3 波动方程

§6.3.1 弦的横振动

考虑一根柔软的弦，平衡时沿 x 轴绷紧。当受到外界扰动时，它可以在垂直于 x 轴的方向作横向振动。弦上的张力使得各部分互相牵制，从而振动可以在弦上传播而形成机械波。假定初始激励（即外界扰动）所引起的振动是小振动（其意义将在以下的讨论中逐步明确），并且弦上各点的位移方向平行，即振动发生在平面内，因而位移是一个标量函数（只有一个分量）。在这些前提下，我们可以用 Newton 运动定律来推导位移所满足的波动方程。记弦上 x 点在 t 时刻的位移为 $u(x, t)$ 。通常在振动平面内将 x 轴绕原点逆时针旋转 90° 作为 u 轴，这与观察的位置有关，所以，位移的正向的规定带有一定的任意性。如果振动发生在竖直平面内，一般取向上为正。波动方程就是 $u(x, t)$ 所满足的微分方程，一般来说是偏微分方程。为清楚起见，下面将推导过程分为几步。

1. 由于弦是柔软的，所以它不抗弯曲，放松时可以取任意形状。在张紧的状态下，不管是处于平衡状态或运动状态，弦上的张力必定沿切线方向。这是由柔软的假定所得出的结论。

2. 考虑运动引起的弦长的变化。平衡时位于区间 $[x, x + \Delta x]$ 的一小段，当运动时，其长度为

$$\Delta s = \int_x^{x+\Delta x} \sqrt{(dx)^2 + (du)^2} = \int_x^{x+\Delta x} \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} dx.$$

由于所考虑的是小振动，故 $|\partial u / \partial x| \ll 1$ ，其二次项可以忽略，故由上式得到

$$\Delta s \approx \int_x^{x+\Delta x} dx = \Delta x.$$

可见运动没有使得弦进一步伸长。由于张力取决于弦的伸长量，所以弦上的张力不随时间变化。这是由小振动的假定所得出的结论。

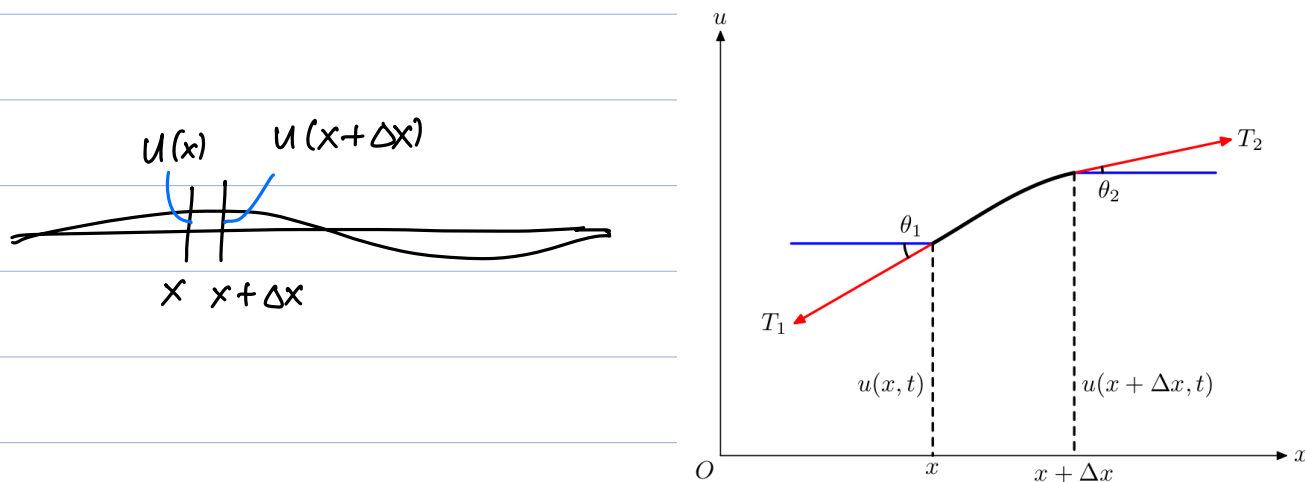


图 1: 弦上一小段的位移和受力分析

3. 考虑弦上平衡时位于区间 $[x, x + \Delta x]$ 的一小段在 x 方向的运动方程. 如图 1 所示, 其中 T_1 是 x 处的张力, T_2 是 $x + \Delta x$ 处的张力, 图中对位移作了夸大. 由于弦上各点都只有 u 方向的位移, 故 x 方向的运动方程为

$$T_2 \cos \theta_2 - T_1 \cos \theta_1 = 0.$$

注意这里用了 Newton 第二定律. 由于所考虑的是小振动, 故 $|\theta_1| \ll 1$ 和 $|\theta_2| \ll 1$, 所以 $\cos \theta_1 \approx 1$ (忽略了二阶小量), $\cos \theta_2 \approx 1$, 于是由上式近似得到

$$T_2 = T_1 \equiv T. \quad (29)$$

这就是说, 弦上的张力不随位置而变化. 这也是由小振动的假定所得出的结论. 综合以上两点可知, 弦上的张力是一个常数.

4. 考虑上述小段在 u 方向的运动方程. 记弦的线密度 (即单位长度的质量) 为 ρ . 由 Newton 第二定律, 有

$$F = ma \Rightarrow \underbrace{\rho \Delta x}_m \underbrace{\frac{\partial^2 \bar{u}}{\partial t^2}}_a = \underbrace{T \sin \theta_2 - T \sin \theta_1}_F, \quad (30)$$

$$\begin{aligned} \rho \Delta x \frac{\partial^2 u}{\partial t^2} &= T \sin \theta_2 - T \sin \theta_1 \\ &\approx T (\tan \theta_2 - \tan \theta_1) \\ &= T \left(\frac{\partial u}{\partial x} \Big|_{x+\Delta x} - \frac{\partial u}{\partial x} \Big|_x \right) \end{aligned}$$

$$\rho \frac{\partial^2 u}{\partial t^2} = T \cdot \frac{u_x(x+\Delta x) - u_x(x)}{\Delta x}$$

$$\Delta x \rightarrow 0 \quad \rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad a = \sqrt{\frac{T}{\rho}}$$

↓
振动在弦上传播的速度

自由振动: 无非齐次项

受迫振动

x 处单位长度受力: $F(x, t)$

$$\rho \Delta x \frac{\partial^2 \bar{u}}{\partial t^2} = T \sin \theta_2 - T \sin \theta_1 + \bar{F} \Delta x$$

$$\Delta x \rightarrow 0$$

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f, \quad f = \frac{F(x, t)}{\rho}$$

$\downarrow$
 x 处单位质量的受力

非齐次项 (inhomogeneous term)

非线性项 (nonlinear)

$$L \cdot u = f$$

$\searrow$ 和 u 无关

$$L = \partial_t^2 - a^2 \partial_x^2 \quad \text{线性微分算符. (linear diff op.)}$$

$$\partial_x^2 u + x \partial_x u + u^2 = f$$

$$(\dots) u = f \quad \text{不可能.}$$

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§6.3.2 杆的纵振动

横波、纵波

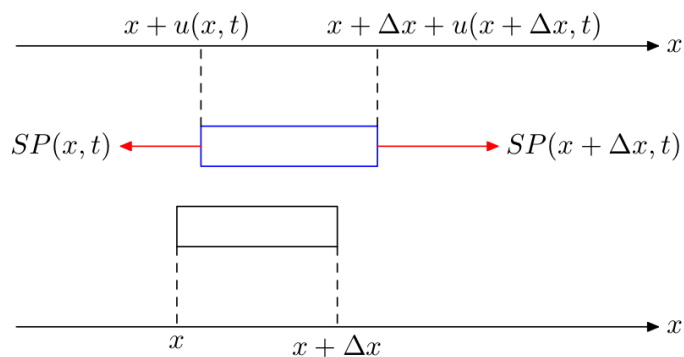


图 2: 杆上一小段. 下图为其平衡位置, 上图为其运动时的位置和受力分析.

杆上一点 x 在 t 时刻的位移为 $u(x, t)$

点 x 的应力为 $P(x, t)$

$$\underbrace{\rho S \Delta x}_m \underbrace{\frac{\partial^2 u}{\partial t^2}}_a = \underbrace{SP(x + \Delta x, t) - SP(x, t)}_F \quad \leftarrow F = ma$$

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{P(x + \Delta x, t) - P(x, t)}{\Delta x} = \frac{\partial P}{\partial x}$$

Hooke 定律:

$$P(x, t) = Y \frac{\partial u(x, t)}{\partial x}$$

↓ 杨氏模量

性质均匀的材料

$$\rho \frac{\partial^2 u}{\partial t^2} = Y \frac{\partial^2 u}{\partial x^2}$$

$$u_{tt} - a^2 u_{xx} = 0, \quad a = \sqrt{\frac{Y}{\rho}}$$

麦氏方程组 $\Rightarrow$ 波动方程

普遍性

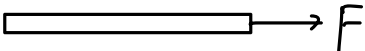
受迫振动

$$\rho S \Delta x \frac{\partial^2 u}{\partial t^2} = SP(x+\Delta x, t) - SP(x, t) + \underline{\bar{F} S \Delta x}$$

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{P(x+\Delta x, t) - P(x, t)}{\Delta x} + \bar{F}$$

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f$$

$$a = \sqrt{\frac{T}{\rho}}, \quad f = \frac{F(x, t)}{\rho}$$

注: 



§6.3.3 定解条件

= 阶 ODE —— 两个初始条件 initial condition

$$\text{波动方程} \quad \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f$$

$$\text{初始条件: } u|_{t=0} = \varphi(x), \quad \frac{\partial u}{\partial t}|_{t=0} = \psi(x) \quad \rightarrow \text{已知函数}$$

给定弦上或杆上各点的初始位移和初始速度

边界条件 boundary condition

PDE vs. ODE

1. 第一类边界条件: 给定边界上的 u 值
(Dirichlet 条件)

弦的横振动, 两个端点固定:



$$u|_{x=0} = 0,$$

a

$$u|_{x=l} = 0$$

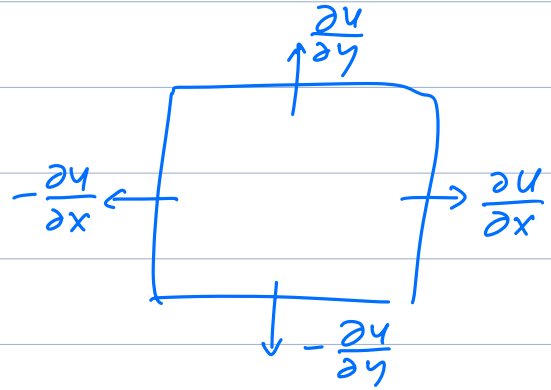
b

齐次

非齐次

2. 第二类边界条件: 给定边界上的 $\frac{\partial u}{\partial n}$ 值, 即外法向的方向导数.
(Neumann 条件)

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} \quad \equiv u_x$$



$$\frac{\partial u}{\partial n} = \vec{n} \cdot \nabla u$$

$$\frac{\partial}{\partial n} \equiv \vec{n} \cdot \nabla$$

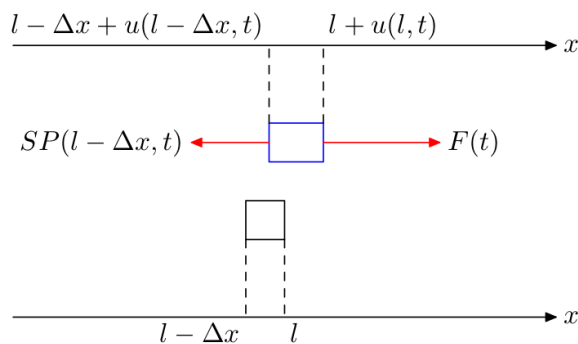


图 3: 第二类边界条件

$$\rho S \Delta x \frac{\partial^2 u}{\partial x^2} = F(t) - SP(l - \Delta x, t)$$

$$\Delta x \rightarrow 0 \quad 0 = F(t) - S \cdot Y \frac{\partial u}{\partial x} \Big|_{x=l}$$

$$\frac{\partial u}{\partial x} \Big|_{x=l} = \frac{1}{YS} F(t)$$

$$F(t) = 0 : \quad \frac{\partial u}{\partial x} \Big|_{x=l} = 0$$

↑ 自由端具有第二类齐次边界条件

3. 第三类边界条件: 给定边界上的 u 和 $\frac{\partial u}{\partial n}$ 的线性组合
(Robin 条件)

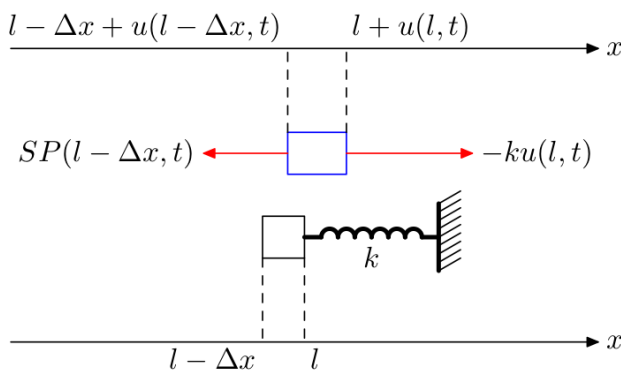


图 4: 第三类边界条件

$$\rho S \Delta x \frac{\partial^2 u}{\partial t^2} = -ku(l, t) - SP(l - \Delta x, t)$$

$$\Delta x \rightarrow 0 \quad 0 = -ku - S \cdot Y \frac{\partial u}{\partial x} \Big|_{x=l}$$

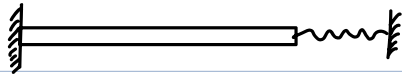
$$\left(\frac{\partial u}{\partial x} + \frac{k}{YS} u \right) \Big|_{x=l} = 0$$

$$\left(\frac{\partial u}{\partial x} - \frac{k}{YS} u \right) \Big|_{x=0} = 0$$

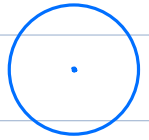
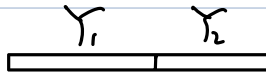
↑ 如果弹簧已经有一定形变: 非齐次

思: $k \rightarrow 0$; $k \rightarrow \infty$

定解问题 $\begin{cases} \text{PDE} \\ \text{定解条件} \end{cases} \begin{cases} \text{初始条件 IC} \\ \text{边界条件 BC} \end{cases}$



连接条件



§ 6.3.4 定解问题的适定性

定解问题的适定性指的是其解的存在性、唯一性和稳定性。存在性指的是有解。唯一性指的是解是确定的，没有任意性。读者对这两点已经比较熟悉。稳定性指的是，当定解条件有微小的变化时，所引起的解的变化也是微小的。这在实际问题中非常重要，因为由测量给出的定解条件与实际情况总会有一定的误差，如果定解问题不具有稳定性，那么理论计算所得到的解将不能反映物体的运动情况，因而是没有实际意义的。

存在性 三等分角

唯一性 电磁学

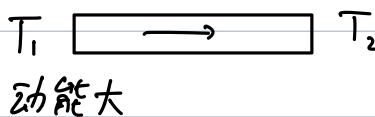
稳定性 混沌

定解条件不能自相矛盾

§ 6.4 热传导方程和扩散方程

§ 6.4.1 热传导方程

热传导



$T_1 > T_2$
不可逆

热流强度：单位时间内垂直流过单位面积的热量

$$\vec{q} = -k \nabla u$$

↑ 温度分布

← 热传导定律

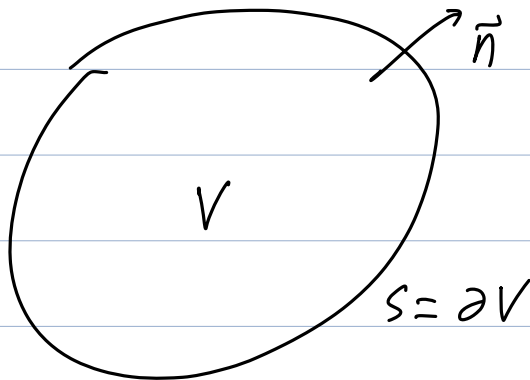
$$\Delta Q = cm \Delta T$$

热流强度 $\frac{1}{\Delta t \Delta S}$

能量密度 $\frac{1}{\Delta V}$

x 流强度 flux

xx 密度 density



能量守恒:

$$\iiint_V c \rho \frac{\partial u}{\partial t} d^3 \vec{r} = - \underbrace{\iint_S \vec{g} \cdot d\vec{\sigma}}_{\text{Gauss 定理}} + \iiint_V \underbrace{F}_{\substack{\text{源 source} \\ \text{汇 sink}}} d^3 \vec{r}$$

" $\parallel$

$$\Rightarrow c \rho \frac{\partial u}{\partial t} - \nabla \cdot (k \nabla u) = F$$

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right)$$

$$\nabla \cdot \vec{A} = \frac{\partial A}{\partial x} + \frac{\partial A}{\partial y} + \frac{\partial A}{\partial z}$$

$$c \rho \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left(k \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left(k \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial z} \left(k \frac{\partial u}{\partial z} \right) = F$$

均匀介质: k, c, ρ 都是常量

$$\frac{\partial u}{\partial t} - a^2 \nabla^2 u = f$$

$$\text{8ftt} \quad \frac{\partial^2 u}{\partial t^2} - a^2 \nabla^2 u = f$$

§6.4.2 扩散方程

扩散流强度 $\vec{q}$

$$\vec{q} = -D \nabla u$$

杂质浓度分布 $u(\vec{r}, t)$

← 扩散定律

物质守恒定律

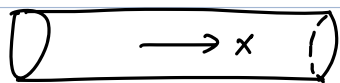
$$\int_V \frac{\partial u}{\partial t} d\vec{r} = - \int_S \vec{q} \cdot d\vec{\sigma} + \int_V F d\vec{r}$$

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{q} = F$$

$$\frac{\partial}{\partial t} - \nabla \cdot (D \nabla u) = F$$

$$\frac{\partial u}{\partial t} - a^2 \nabla^2 u = f$$

$$a = \sqrt{D}$$



微观机制：热传导、扩散都是通过分子碰撞完成的——输运

§6.4.3 定解条件

初始条件：初始时刻的温度分布或浓度分布

$$u|_{t=0} = \varphi(\vec{r}), \quad \vec{r} \in \Omega$$

↑

边界条件

1. 第一类边界条件: 将研究对象置于温度已知的环境中

$$u|_{\partial\Omega} = u_0(\vec{r}, t), \quad \vec{r} \in \partial\Omega$$

↓ 环境温度

2. 第二类边界条件: 在边界面上有已知的热流流入

$$\frac{\partial u}{\partial n}|_{\partial\Omega} = \frac{1}{k} q(\vec{r}, t), \quad \vec{r} \in \partial\Omega$$

$\vec{n} \cdot \nabla u$

$$q(\vec{r}, t) = 0: \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0$$

绝热的边界面: 第二类齐次边界条件

对应弦: 端点自由

3. 第三类边界条件: 边界面按牛顿冷却定律与外界交换热量

$$q_n|_{\partial\Omega} = b[u|_{\partial\Omega} - u_0(\vec{r}, t)] \quad \vec{r} \in \partial\Omega$$

$$q_n = \vec{q} \cdot \vec{n}$$



热传导定律: $q_n|_{\partial\Omega} = -k \frac{\partial u}{\partial n}|_{\partial\Omega}$

$$\left(\frac{\partial u}{\partial n} + hu\right)|_{\partial\Omega} = hu_0(\vec{r}, t) \quad \vec{r} \in \partial\Omega$$

$$h = \frac{b}{k}$$

- 细杆 & $u_0 = 0$

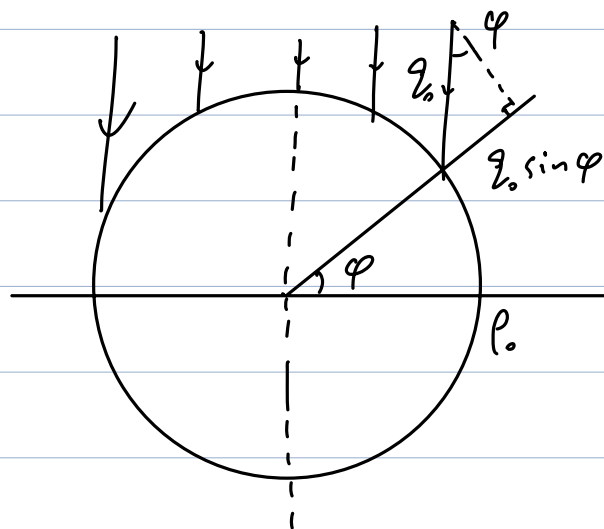
$$\left(\frac{\partial u}{\partial x} - hu\right)|_{x=0} = 0$$

$$\left(\frac{\partial u}{\partial x} + hu\right)|_{x=l} = 0$$

例：半径为 ρ_0 而表面熏黑的金属长圆柱，受阳光照射。

阳光方向垂直于柱轴，热流强度为 q_0 ，周围介质的温度为 θ 。

写出这个圆柱的热传导问题的边界条件。



散热 $b(u - \theta)$

黑体. 模型语言. 如质点, 惯性系

太阳可以看作黑体

牛顿冷却定律

$$-k u_r|_{\rho=\rho_0} + q_0 \sin \varphi = b(u - \theta)|_{\rho=\rho_0}$$

$$(k u_r + b(u - \theta))|_{\rho=\rho_0} = \begin{cases} q_0 \sin \varphi & 0 < \varphi < \pi \\ 0 & \pi \leq \varphi \leq 2\pi \end{cases}$$

§6.5 稳定场方程

§6.5.1 稳定温度分布和稳定浓度分布

$$\frac{\partial u}{\partial t} - a^2 \nabla^2 u = \underline{f(\vec{r})}$$

↓

$$\nabla^2 u = -\frac{f}{a^2}$$

Poisson 方程

Laplace 方程 无源

注: f 和 BC 与 t 无关只是达到稳定温度分布的必要条件
稳定状态不一定是平衡状态

$$u_1 \quad \boxed{} \quad u_2$$

§6.5.2 静电场方程

$$\text{介质中 } \nabla \cdot \vec{D} = \rho, \quad \nabla \times \vec{E} = 0$$
$$\vec{E} = -\nabla u$$

线性、各向同性的均匀介质: $\vec{D}(\vec{r}) = \epsilon \vec{E}(\vec{r})$

$$\nabla^2 u = -\frac{\rho}{\epsilon} \quad \leftarrow \text{自由电荷所在处}$$

§6.5.3 定解条件

三种方程

$$(\partial_t^2 - a^2 \nabla^2) u = f(\vec{r}, t)$$

$$u_{xx} - a^2 u_{xx} = f(x, t)$$

$$(\partial_t - a^2 \nabla^2) u = f(\vec{r}, t)$$

$$\nabla^2 u = f(\vec{r})$$

三类边界条件

第一类: $u(\vec{r}, t)|_{\Sigma} = f(M, t)$

第二类: $\frac{\partial u}{\partial n}|_{\Sigma} = f(M, t)$

第三类: $(u + H \frac{\partial u}{\partial n})|_{\Sigma} = f(M, t)$

Chap. 7 分离变量法

PDE $\rightarrow$ 若干 ODE

$$y'' + k^2 y = 0 \Rightarrow y = C_1 \cos kx + C_2 \sin kx$$

$$y'' - \mu^2 y = 0 \Rightarrow \begin{cases} y = C_1 \cosh \mu x + C_2 \sinh \mu x \\ y = C_1 e^{\mu x} + C_2 e^{-\mu x} \end{cases}$$

§7.1 一维波动方程

§7.1.1 分离变量

定解问题:

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 & (0 < x < l, t > 0) \\ u|_{x=0} = 0, \quad u|_{x=l} = 0 & (t \geq 0) \quad \text{B.C.} \\ u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x) & (0 \leq x \leq l) \quad \text{I.C.} \end{cases}$$

$\swarrow \searrow$
已知函数

尝试寻找下列形式的特解:

$$u(x, t) = X(x) T(t)$$

代入方程得

$$X(x) T''(t) - a^2 X''(x) T(t) = 0$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = -\lambda$$

$\downarrow$
与 t 无关

$\downarrow$
与 x 无关

$F(x) = G(t)$ 对 x, t 恒成立 $\Rightarrow$ const.

$$\Rightarrow T''(t) + \lambda a^2 T(t) = 0$$

$$X''(x) + \lambda X(x) = 0$$

$$\text{Let } \lambda \in \mathbb{R}. \quad \mathcal{Z}(0)T(t) = \mathcal{Z}(l)T(t) = 0 \quad (*)$$

$$\text{I.C.} \quad \mathcal{Z}(x)T(0) = \varphi(x), \quad \mathcal{Z}(x)T'(0) = \psi(x) \quad (*)'$$

(*) : $T(t)$ 不能恒为零 (非平庸解) $\Rightarrow$

$$\mathcal{Z}(0) = 0, \quad \mathcal{Z}(l) = 0$$

(*)' : $\varphi(x) \propto \mathcal{Z}(x)$, $\psi(x) \propto \mathcal{Z}(x)$, 且 $\varphi(x)$ 和 $\psi(x)$ 相当 (任意).

§ 1.2 本征值问题

$$\begin{cases} \mathcal{Z}''(x) + \lambda \mathcal{Z}(x) = 0 & \textcircled{1} \end{cases}$$

$$\begin{cases} \mathcal{Z}(0) = 0, \quad \mathcal{Z}(l) = 0 & \textcircled{2} \end{cases}$$

1. $\lambda = 0$ ① solution $\mathcal{Z}(x) = Cx + D$

Let $\lambda \textcircled{2}$, $C = D = 0 \Rightarrow \mathcal{Z}(x) \equiv 0$ 平庸解

2. $\lambda < 0$, $\hat{=} \lambda = -\mu^2$ ($\mu > 0$)

① solution: $\mathcal{Z}(x) = C \cosh \mu x + D \sinh \mu x$

$$\mathcal{Z}(0) = 0 \Rightarrow C = 0$$

$$\mathcal{Z}(l) = 0 \Rightarrow D \sinh \mu l = 0 \Rightarrow D = 0 \text{ 平庸解}$$

3. $\lambda > 0$ $\hat{=} \lambda = \mu^2$ ($\mu > 0$)

① solution: $\mathcal{Z}(x) = C \cos \mu x + D \sin \mu x$

$$\mathcal{Z}(0) = 0 \Rightarrow C = 0$$

$$\mathcal{Z}(l) = 0 \Rightarrow D \sin \mu l = 0$$

非平庸解: $\sin \mu l = 0 \Rightarrow \mu_n = \frac{n\pi}{l} \quad (n \in \mathbb{N}^+)$

$\lambda_n = \left(\frac{n\pi}{l}\right)^2$ 本征值

$\mathcal{B}_n(x) = \sin \frac{n\pi}{l} x$ 本征函数

[已取 $D=1$. $T_n(t)$ 中会包含积分常数.]

$u_n(x, t) = \mathcal{B}_n(x) T_n(t)$

或 2. $\lambda > 0$

$\mathcal{B}(x) = C \cos \sqrt{\lambda} x + D \sin \sqrt{\lambda} x$

$\mathcal{B}(0) = 0 \Rightarrow C = 0$

$\mathcal{B}(l) = 0 \Rightarrow D \sin \sqrt{\lambda} l = 0 \xrightarrow{D \neq 0} \sqrt{\lambda} l = n\pi$
 $\Rightarrow \lambda_n = \left(\frac{n\pi}{l}\right)^2, n \in \mathbb{N}^+$

3. $\lambda < 0$

$\mathcal{B}(x) = C \cosh \sqrt{|\lambda|} x + D \sinh \sqrt{|\lambda|} x$

$\mathcal{B}(0) = 0 \Rightarrow C = 0$

$\mathcal{B}(l) = 0 \Rightarrow D \sinh \sqrt{|\lambda|} l = 0 \Rightarrow D = 0 \Rightarrow \mathcal{B} \equiv 0$

§7.1.3 本征函数族的正交完备性

$\left\{ \mathcal{B}_n(x) = \sin \frac{n\pi x}{l} \right\}_{n=1}^{\infty}$

在 $[0, l]$ 正交

$(\mathcal{B}_m(x), \mathcal{B}_n(x)) = \int_0^l \mathcal{B}_m(x) \mathcal{B}_n(x) dx = \int_0^l \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} dx = 0$
 (内积) ($m \neq n$)

完备性

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x, \quad x \in [0, l]$$

$$a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x dx, \quad n \in \mathbb{N}^+$$

$$\|g_n(x)\|^2 \equiv (g_n(x), g_n(x)) = \int_0^l \sin^2 \frac{n\pi}{l} x dx = \frac{l}{2}$$

$$(g_m(x), g_n(x)) = \int_0^l \sin \frac{m\pi}{l} x \sin \frac{n\pi}{l} x dx = \frac{l}{2} \delta_{mn}, \quad \forall m, n \in \mathbb{N}^+$$

$$\delta_{mn} = \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases}$$

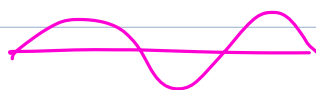
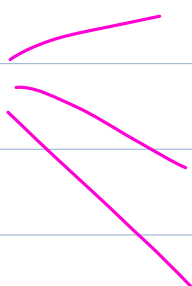
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$$\begin{cases} x+y=1 \\ 3x-y=2 \end{cases} \Rightarrow \underbrace{\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}}_M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{cases} x+y=0 \\ 3x-y=0 \end{cases} \quad \text{无解}$$

$$M\gamma = \lambda\gamma \Rightarrow (M - \lambda I)\gamma = 0 \quad \Rightarrow \det |M - \lambda I| = 0$$

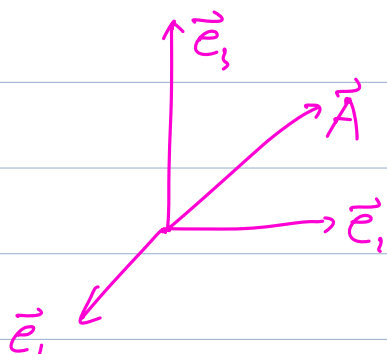
$\downarrow$
齐次



⋮



类比:



$$\vec{A} = A_1 \vec{e}_1 + A_2 \vec{e}_2 + A_3 \vec{e}_3 \\ = \sum_{i=1}^3 A_i \vec{e}_i$$

$$\vec{e}_i \cdot \vec{e}_j = 0, \quad \vec{e}_i \cdot \vec{e}_i = 1, \dots \\ \text{合并: } \vec{e}_i \cdot \vec{e}_j = \delta_{ij}$$

$$A_1 = \vec{A} \cdot \vec{e}_1, \quad A_i = \vec{A} \cdot \vec{e}_i$$

类比:

$$MA = B$$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

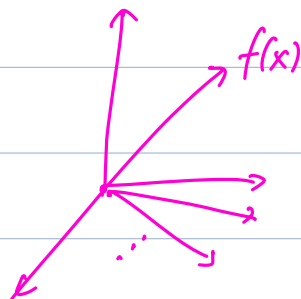
本征值

$$My = \lambda y$$

$$(M - \lambda I)y = 0$$

$$\det |M - \lambda I| = 0$$

$\Rightarrow$ 本征值 (n个)



$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x$$

$$\int_0^l \sin \frac{m\pi}{l} x \sin \frac{n\pi}{l} x dx = \frac{l}{2} \delta_{mn}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$L = \partial_x^2 + \lambda$$

$$\begin{matrix} x=0 \rightarrow \\ x=l \rightarrow \end{matrix} \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$



$$y(0) = 0$$

$$y(l) = 0$$

齐次方程, 边界条件

$\Rightarrow$ 本征值 (n个)

§7.1.4 - 一般解

$$T'' + \lambda a^2 T(t) = 0$$

$$T_n''(t) + \left(\frac{n\pi a}{l}\right)^2 T_n(t) = 0$$

$$\Rightarrow T_n = A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t$$

$$u_n(x, t) = X_n(x) T_n(t) = \left(A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t \right) \sin \frac{n\pi}{l} x$$

$n \in \mathbb{N}^+$

这些解一般不满足 I.C.

上述解叠加后仍然满足方程和边界条件

一般解:

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t \right) \sin \frac{n\pi}{l} x$$

注: (1) 一般解不具有分离变量的形式

(2) 一般解不是通解

PDE + BC

(3) $\varphi(x)$ 和 $\psi(x)$ 的解析性质

§7.1.5 用初始条件确定系数

将一般解 $u(x, t)$ 代入初始条件得

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{l} x = \varphi(x), \quad \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin \frac{n\pi}{l} x = \psi(x)$$

$$A_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi}{l} x \, dx, \quad B_n = \frac{2}{n\pi a} \int_0^l \psi(x) \sin \frac{n\pi}{l} x \, dx$$

$$\vec{A}_i = \frac{1}{\|\vec{e}_i\|^2} \vec{A} \cdot \vec{e}_i$$

分离变量法, Fourier 方法, 驻波法

§7.1.6 解的物理图像

$$\begin{aligned} u_n(x,t) &= \left(A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t \right) \sin \frac{n\pi}{l} x \\ &= \sqrt{A_n^2 + B_n^2} \cos \left(\frac{n\pi a}{l} t + \delta_n \right) \sin \frac{n\pi}{l} x \\ &= C_n \sin k_n x \cos(\omega_n t + \delta_n) \end{aligned}$$

ω_n 本征频率

基波, n 次谐波

$$k_n x = m\pi \Rightarrow x = \frac{m}{n} l \quad m = 0, 1, \dots, n$$

波节

弦上的一般振动是各次谐波的线性叠加

§7.2 ~~波动方程~~ 一维热传导方程

§7.2.1 分离变量与本征值问题

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (0 < x < l, t > 0)$$

$$\frac{\partial u}{\partial x} \Big|_{x=0} = 0, \quad \frac{\partial u}{\partial x} \Big|_{x=l} = 0 \quad (t \geq 0)$$

$$u|_{t=0} = \varphi(x), \quad \cancel{u|_{t=0} = \psi(x)} \quad (0 \leq x \leq l)$$

$$u(x,t) = X(x)T(t)$$

$$\begin{cases} Z''(x) + \lambda Z(x) = 0 \\ Z'(0) = 0, \quad Z'(l) = 0 \end{cases}$$

$$T''(t) + \lambda a^2 T(t) = 0$$

1. $\lambda < 0, \quad \lambda = -\mu^2$

$$Z = C \cosh \mu x + D \sinh \mu x$$

$$Z' = -\mu C \sinh \mu x + \mu D \cosh \mu x$$

$$Z'(0) = 0 \Rightarrow D = 0$$

$$Z'(l) = 0 \Rightarrow C \sinh \mu l = 0 \Rightarrow C = 0$$

2. $\lambda = 0$: $Z = Cx + D$

$$Z' = C$$

$$Z'(0) = 0 \Rightarrow C = 0$$

$$Z'(l) = 0 \Rightarrow$$

$$\lambda_0 = 0, \quad Z_0(x) = 1$$

3. $\lambda > 0, \quad \lambda = \mu^2$

$$Z = C \cos \mu x + D \sin \mu x$$

$$Z' = -\mu C \sin \mu x + \mu D \cos \mu x$$

$$Z'(0) = 0 \Rightarrow D = 0$$

$$Z'(l) = 0 \Rightarrow C \sin \mu l = 0 \Rightarrow \mu l = n\pi \Rightarrow \mu_n = \frac{n\pi}{l}$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad Z_n(x) = \cos \frac{n\pi}{l} x \quad n \in \mathbb{N}^+$$

所有本征值和本征函数

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad Z_n(x) = \cos \frac{n\pi}{l} x \quad n \in \mathbb{N}$$

§7.2.2 本征函数族的正交完备性

$$\left\{ X_n(x) = \cos \frac{n\pi}{l} x \right\}_{n=0}^{\infty}$$

$$\left\{ 1, \cos \frac{n\pi}{l} x \right\}$$

$$(X_m(x), X_n(x)) = \int_0^l \cos \frac{m\pi}{l} x \cos \frac{n\pi}{l} x dx = 0, \quad m, n \in \mathbb{N}^+, \quad m \neq n,$$

> 正交

$$(X_0(x), X_n(x)) = \int_0^l \cos \frac{n\pi}{l} x dx = 0, \quad n \in \mathbb{N}^+.$$

$$\|X_n(x)\|^2 \equiv (X_n(x), X_n(x)) = \int_0^l \cos^2 \frac{n\pi}{l} x dx = \begin{cases} l, & n = 0; \\ l/2, & n \neq 0. \end{cases} \quad \leftarrow (1,1) = \int_0^l dx = l$$

$$f(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi}{l} x = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l} x, \quad x \in [0, l],$$

$$\vec{A} = \sum_{i=1}^3 A_i \vec{e}_i$$

其中的展开系数为

$$a_0 = \frac{1}{l} \int_0^l f(x) dx, \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi}{l} x dx, \quad n \in \mathbb{N}^+.$$

$$\vec{A}_i = \frac{1}{\|\vec{e}_i\|^2} \vec{A} \cdot \vec{e}_i$$

§7.2.3 一般解

$$T''(t) + \lambda_n a^2 T(t) = 0$$

$$u(x,t) = \underbrace{A_0 t + B_0}_{\text{质心运动}} + \sum_{n=1}^{\infty} \left(\cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t \right) \cos \frac{n\pi}{l} x$$

$$T'(t) + \lambda_n a^2 T(t) = 0$$

$$T_0(t) = 0, \quad T_n(t) = A_n e^{-\lambda_n a^2 t} = A_n e^{-\left(\frac{n\pi a}{l}\right)^2 t}$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n a^2 t} \cos \frac{n\pi}{l} x$$

§7.2.4 用初始条件确定系数

$$A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{l} x = \varphi(x)$$

$$A_0 = \frac{1}{l} \int_0^l \varphi(x) dx \quad A_n = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{n\pi}{l} x dx$$

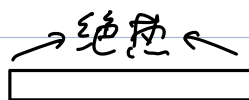
如果 $\varphi(x)$ 已经写成 $\cos \frac{n\pi}{l} x$, 无需算积分

$$\varphi(x) = 1 + 2 \cos \frac{2\pi}{l} x$$

§7.2.5 解的物理图象

$$t \rightarrow \infty$$

$$u(x, t) \rightarrow A_0 = \frac{1}{l} \int_0^l \varphi(x) dx$$



初始温度的平均值

§7.3 非齐次方程 与非齐次边界条件

$$\begin{cases} \text{齐次方程} \\ \text{齐次边界条件} \end{cases} \Rightarrow u(x, t) = \sum_n T_n(t) \underbrace{\delta_n(x)}_{\text{本征函数}}$$

$$\begin{cases} \text{方程} \\ \text{非齐次边界条件} \end{cases} \xrightarrow{\text{齐次化}} \begin{cases} \text{(非齐次) 方程} \\ \text{齐次边界条件} \end{cases}$$

化归

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 \\ u|_{x=0} = 1, \quad u|_{x=l} = 1 \\ u|_{t=0} = \psi(x), \quad u_t|_{t=0} = \varphi(x) \end{cases}$$

$$u(x, t) = v(x, t) + 1$$

非齐次方程 $y'' + k^2 y = f(x)$

§7.3.1 非齐次边界条件的齐次化

$$\begin{cases} U_{tt} - a^2 U_{xx} = 0 & (0 < x < l) \\ U|_{x=0} = 0, \quad U|_{x=l} = A \sin \omega t & (t \geq 0) \\ U|_{t=0} = 0, \quad U_t|_{x=l} = 0 & (0 \leq x \leq l) \end{cases}$$

$$U(x, t) = V(x, t) + U_0(x, t)$$

↑ 与 $U(x, t)$ 满足相同的 BC, 尽可能简单

$$U_0 = \frac{x}{l} A \sin \omega t$$

$$U_x = V_x + \frac{1}{l} A \sin \omega t, \quad U_{xx} = V_{xx}$$

$$U_t = V_t + \frac{x}{l} A \omega \cos \omega t, \quad U_{tt} = V_{tt} - \frac{x}{l} A \omega^2 \sin \omega t$$

$$V|_{x=0} + 0 = 0, \quad V|_{x=l} + A \sin \omega t = A \sin \omega t$$

$$V|_{t=0} + 0 = 0, \quad V_t|_{t=0} + \frac{x}{l} A \omega = 0$$

$$\begin{cases} V_{xx} - a^2 V_{tt} = \frac{\omega^2 A}{l} x \sin \omega t \\ V|_{x=0} = 0, \quad V|_{x=l} = 0 \\ V|_{t=0} = 0, \quad V_t|_{t=0} = -\frac{\omega A}{l} x \end{cases}$$

非齐次方程

齐次边界条件

$$U_0(x, t) = A \frac{\sin \frac{\omega x}{a}}{\sin \frac{\omega l}{a}} \sin \omega t$$

$$\begin{cases} V_{tt} - a^2 V_{xx} = 0 \\ V|_{x=0} = 0, \quad V|_{x=l} = 0 \\ V|_{t=0} = 0, \quad V_t|_{t=0} = -\omega A \frac{\sin \frac{\omega x}{a}}{\sin \frac{\omega l}{a}} \end{cases}$$

一般解: $V(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t \right) \sin \frac{n\pi}{l} x$

代入 IC

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{l} x = 0, \quad \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin \frac{n\pi}{l} x = -\omega A \frac{\sin \frac{\omega x}{a}}{\sin \frac{\omega l}{a}}$$

$$A_n = 0$$

$$B_n = \frac{2\omega A}{l a} \frac{(-1)^{n+1}}{\left(\frac{\omega}{a}\right)^2 - \left(\frac{n\pi}{l}\right)^2}$$

选择 $u_0(x,t)$

例:
$$\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t) \\ u|_{x=0} = \bar{u}(t), \quad u|_{x=l} = \bar{v}(t) \\ u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x) \end{cases}$$

$$u(x,t) = v(x,t) + \bar{u}(t) + \frac{x}{l} [\bar{v}(t) - \bar{u}(t)]$$

$$\begin{cases} v_{tt} - a^2 v_{xx} = f(x,t) - \bar{u}''(t) - \frac{x}{l} [\bar{v}''(t) - \bar{u}''(t)] \\ v|_{x=0} = 0, \quad v|_{x=l} = 0 \\ v|_{t=0} = \varphi(x) - \bar{u}(0) - \frac{\bar{v}(0) - \bar{u}(0)}{l} x, \quad v_t|_{t=0} = \psi(x) - \dots \end{cases}$$

$u_0(x,t)$ 的选择不唯一, 满足边界条件即可

§7.3.2 非齐次方程的本征函数展开法

假设边界条件已经齐次化, 则弦振动方程第一边值问题的一般形式是

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x, t), \quad (0 < x < l, t > 0) \quad (67a)$$

$$u|_{x=0} = 0, \quad u|_{x=l} = 0, \quad (t \geq 0) \quad (67b)$$

$$u|_{t=0} = \varphi(x), \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(x). \quad (0 \leq x \leq l) \quad (67c)$$

事实上, 式 (65) 就具有这样的形式. 这也可以看作两端固定的弦的受迫振动问题. 现在考虑这一定解问题的求解.

$u(x, t)$ 是 x 和 t 两个变量的函数, 如果固定时刻 t , 那么它就只是 x 的函数, 这时它可以展开为某一完备的本征函数族的线性组合, 比如前两节中出现的本征函数族 $\{\sin(n\pi x/l)\}_{n=1}^{\infty}$ 或 $\{\cos(n\pi x/l)\}_{n=0}^{\infty}$, 当时刻 t 变化时, 展开系数也就相应地不同, 所以在一般时刻 t 也应该可以作这样的展开, 只不过展开系数是 t 的函数, 即

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi}{l} x, \quad (68)$$

用相应的齐次方程的定解问题中所解出的本征函数族来展开 u

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi}{l} x, \quad f_n(t) = \frac{2}{l} \int_0^l f(x, t) \sin \frac{n\pi}{l} x dx, \quad n \in \mathbb{N}^+;$$

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n \sin \frac{n\pi}{l} x, \quad \varphi_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi}{l} x dx, \quad n \in \mathbb{N}^+;$$

$$\vec{A} = \sum_i A_i \vec{e}_i$$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin \frac{n\pi}{l} x, \quad \psi_n = \frac{2}{l} \int_0^l \psi(x) \sin \frac{n\pi}{l} x dx, \quad n \in \mathbb{N}^+.$$

这里各展开系数 $f_n(t)$ 、 φ_n 和 ψ_n 都是已知的, 给定各函数 $f(x, t)$ 、 $\varphi(x)$ 和 $\psi(x)$ 的具体形式, 原则上就可以通过积分将它们计算出来. 将这些展开式和式 (68) 一起代入方程 (67a) 和初始条件 (67c), 可得

$$\sum_{n=1}^{\infty} \left[T_n''(t) + \left(\frac{n\pi a}{l} \right)^2 T_n(t) \right] \sin \frac{n\pi}{l} x = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi}{l} x,$$

$$\vec{A} = \vec{B}$$

$$\sum_{n=1}^{\infty} T_n(0) \sin \frac{n\pi}{l} x = \sum_{n=1}^{\infty} \varphi_n \sin \frac{n\pi}{l} x,$$

$$\sum_i A_i \vec{e}_i = \sum_i B_i \vec{e}_i$$

$$\sum_{n=1}^{\infty} T_n'(0) \sin \frac{n\pi}{l} x = \sum_{n=1}^{\infty} \psi_n \sin \frac{n\pi}{l} x.$$

$$\Rightarrow A_i = B_i$$

$$T_n''(t) + \left(\frac{n\pi a}{l}\right)^2 T_n(t) = f_n(t), \quad (72a)$$

$$T_n(0) = \varphi_n, \quad T_n'(0) = \psi_n. \quad (72b)$$

这是一系列常微分方程的初值问题，按照常微分方程的解法，可以解得

$$T_n(t) = \varphi_n \cos \frac{n\pi a}{l} t + \frac{l\psi_n}{n\pi a} \sin \frac{n\pi a}{l} t + \frac{l}{n\pi a} \int_0^t f_n(\tau) \sin \left[\frac{n\pi a}{l} (t - \tau) \right] d\tau. \quad (73)$$

$$\frac{d}{dx} \leftrightarrow \int dx$$

§ 7.3.3 * 本征函数族的选择

§ 7.3.4 边界条件齐次化的一般方法

$$\left(\alpha \frac{\partial u}{\partial x} - \beta u\right)\Big|_{x=0} = \bar{u}(t), \quad \left(\gamma \frac{\partial u}{\partial x} + \delta u\right)\Big|_{x=l} = \bar{v}(t)$$

$$u(x, t) = v(x, t) + u_0(x, t)$$

$$u_0(x, t) = A(t) + x B(t) \quad \text{或} \quad x A(t) + x^2 B(t)$$

$$\text{例: } u|_{x=0} = \bar{u}(t), \quad u|_{x=l} = \bar{v}(t)$$

$$u_0 = \bar{u} + \frac{x}{l} [\bar{v}(t) - \bar{u}(t)]$$

$$\text{例: } u_x|_{x=0} = \bar{u}(t), \quad u_x|_{x=l} = \bar{v}(t)$$

$$u_0 = x \bar{u}(t) + \frac{x^2}{2l} [\bar{v}(t) - \bar{u}(t)]$$

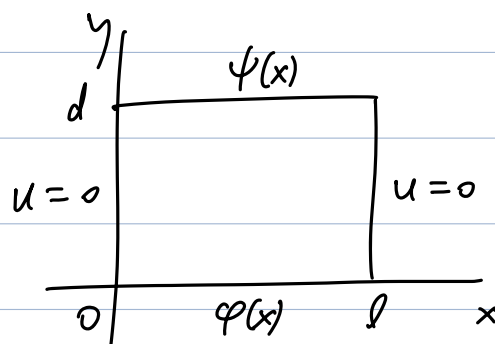
$$\text{例: } u|_{x=0} = \bar{u}(t), \quad u_x|_{x=l} = \bar{v}(t)$$

$$u_0 = \bar{u}(t) + x \bar{v}(t)$$

§4 * 矩形区域上的稳定场方程

只有边界条件, 没有初始条件

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & (0 < x < l, 0 < y < d) \\ u|_{x=0} = 0, \quad \frac{\partial u}{\partial x}|_{x=l} = 0, & (0 \leq y \leq d) \\ u|_{y=0} = \varphi(x), \quad u|_{y=d} = \psi(x), & (0 \leq x \leq l) \end{cases}$$



$$u(x, y) = X(x)Y(y)$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0, \quad X'(l) = 0 \end{cases}$$

$$Y'' - \lambda Y(y) = 0$$

本征值问题

$$(1) \lambda = 0, \quad X = Cx + D$$

$$X(0) = 0 \Rightarrow D = 0$$

$$X(l) = 0 \Rightarrow C = 0 \quad \text{平庸解}$$

$$(2) \lambda > 0 \quad X = C \cos \sqrt{\lambda} x + D \sin \sqrt{\lambda} x$$

$$X(0) = 0 \Rightarrow C = 0$$

$$X'(l) = 0 \Rightarrow \sqrt{\lambda} D \cos \sqrt{\lambda} l = 0$$

$$\sqrt{\lambda} l = (n - \frac{1}{2})\pi, \quad n \in \mathbb{N}^+$$

$$\lambda_n = \left(\frac{(2n-1)\pi}{2l} \right)^2$$

$$\mathcal{Z}_n(x) = \sin \frac{2n-1}{2l} \pi x \quad n \in \mathbb{N}^+$$

$$(3) \quad \lambda < 0 \quad \mathcal{Z} = C \cosh \sqrt{|\lambda|} x + D \sinh \sqrt{|\lambda|} x$$

$$\mathcal{Z}(0) = 0 \Rightarrow C = 0$$

$$\mathcal{Z}(l) = 0 \Rightarrow \sqrt{|\lambda|} D \cosh \sqrt{|\lambda|} l = 0 \Rightarrow D = 0 \quad \text{平凡解}$$

$$\int_0^l \sin \frac{2m-1}{2l} \pi x \sin \frac{2n-1}{2l} \pi x \, dx = \frac{l}{2} \delta_{mn}, \quad m, n \in \mathbb{N}^+$$

$$u(x, y) = \sum_{n=1}^{\infty} Y_n(y) \mathcal{Z}_n(x)$$

$$Y_n'' - \lambda_n Y_n = 0, \quad n \in \mathbb{N}^+$$

$$Y_n = \{ e^{\sqrt{\lambda_n} y}, e^{-\sqrt{\lambda_n} y} \} \quad \text{或} \quad \{ \cosh \sqrt{\lambda_n} y, \sinh \sqrt{\lambda_n} y \}$$

$$Y_n = A_n \sinh \sqrt{\lambda_n} (d-y) + B_n \sinh \sqrt{\lambda_n} y$$

$$u(x, y) = \sum_{n=1}^{\infty} (A_n \sinh \sqrt{\lambda_n} (d-y) + B_n \sinh \sqrt{\lambda_n} y) \sin \sqrt{\lambda_n} x$$

$$\text{代入边界条件 } u|_{y=0}, \quad u|_{y=d} :$$

$$\sum_{n=1}^{\infty} A_n \sinh \sqrt{\lambda_n} d \sin \sqrt{\lambda_n} x = \varphi(x)$$

$$\sum_{n=1}^{\infty} B_n \sinh \sqrt{\lambda_n} d \sin \sqrt{\lambda_n} x = \psi(x)$$

$$A_n \sinh \sqrt{\lambda_n} d = \frac{2}{l} \int_0^l \varphi(x) \sin \sqrt{\lambda_n} x \, dx$$

$$B_n \sinh \sqrt{\lambda_n} d = \frac{2}{l} \int_0^l \psi(x) \sin \sqrt{\lambda_n} x \, dx$$

$$\begin{cases} u_{xx} + u_{yy} = f(x, y) \\ u|_{x=0} = a(y), \quad u|_{x=l} = b(y) \\ u_y|_{y=0} = \varphi(x), \quad u_y|_{y=d} = \psi(x) \end{cases}$$

解法1: 边界条件齐次化

$$u = v + u_0 = v + b + \frac{x}{l}(b-a)$$

关于 $v(x, y)$ 的方程: 非齐次方程, 齐次边界条件 ($x=0, l$)
本征函数展开法

解法2: $u = u_1 + u_2$

$$\begin{cases} \partial_x^2 u_1 + \partial_y^2 u_1 = f(x, y) \\ u_1|_{x=0} = 0, \quad u_1|_{x=l} = 0 \\ \partial_y u_1|_{y=0} = \varphi(x), \quad \partial_y u_1|_{y=d} = \psi(x) \end{cases}$$

$$\begin{cases} \partial_x^2 u_2 + \partial_y^2 u_2 = 0 \\ u_2|_{x=0} = a(y), \quad u_2|_{x=l} = b(y) \\ \partial_y u_2|_{y=0} = 0, \quad \partial_y u_2|_{y=d} = 0 \end{cases}$$

§7.5 平面极坐标系中的稳定场方程

边界形状

坐标面 如球面 $r = \text{const.}$

分离变量 $\rightarrow$ ODE $\rightarrow$ 特殊函数

§7.5.1 一般解

$$\nabla^2 u = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2}$$

$$\nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u$$

$$\begin{cases} x = \rho \cos \varphi \\ y = \rho \sin \varphi \end{cases}$$

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial \rho}{\partial x} \frac{\partial}{\partial \rho} + \frac{\partial \varphi}{\partial x} \frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial y} &= \dots \end{aligned}$$

$$u(\rho, \varphi) = R(\rho) \Phi(\varphi)$$

$$\frac{1}{\rho} (\rho R')' \cdot \Phi + \frac{1}{\rho^2} R \cdot \Phi'' = 0$$

$$\frac{\rho}{R} (\rho R')' = - \frac{\Phi''}{\Phi} = \lambda$$

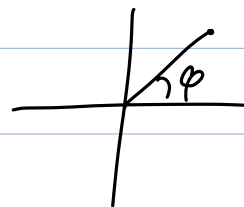
$$\rho^2 R'' + \rho R' - \lambda R = 0$$

$$\Phi'' + \lambda \Phi = 0$$

边界条件

$$u(\rho, \varphi + 2\pi) = u(\rho, \varphi)$$

单值性



$$\Phi(\varphi + 2\pi) = \Phi(\varphi)$$

自然边界条件

周期性边界条件

$$\begin{cases} \Phi'' + \lambda \Phi = 0 \\ \Phi(\varphi + 2\pi) = \Phi(\varphi) \end{cases}$$

$$1. \lambda < 0, \quad \hat{\lambda} = -\mu^2 \quad (\mu > 0)$$

$$\Phi(\phi) = Ce^{\mu\phi} + De^{-\mu\phi}$$

$$Ce^{\mu\phi} + De^{-\mu\phi} = Ce^{\mu(\phi+2\pi)} + De^{-\mu(\phi+2\pi)}$$

$$\Phi(0) = \Phi(2\pi) \Rightarrow C + D = Ce^{\mu \cdot 2\pi} + De^{-\mu \cdot 2\pi}$$

$$\Phi'(0) = \Phi'(2\pi) \Rightarrow \mu(C - D) = \mu(Ce^{\mu \cdot 2\pi} - De^{-\mu \cdot 2\pi})$$

$$\Rightarrow C(e^{\mu \cdot 2\pi} - 1) = 0, \quad D(e^{-\mu \cdot 2\pi} - 1) = 0 \Rightarrow C = D = 0$$

$$2. \lambda = 0, \quad \Phi = C + D\phi$$

$$\lambda_0 = 0, \quad \Phi_0(\phi) = 1$$

$$3. \lambda > 0, \quad \hat{\lambda} = \mu^2 \quad (\mu > 0)$$

$$\Phi(\phi) = Ce^{i\mu\phi} + De^{-i\mu\phi}$$

$$\Phi(0) = \Phi(2\pi) \Rightarrow C + D = Ce^{i\mu \cdot 2\pi} + De^{-i\mu \cdot 2\pi}$$

$$\Phi'(0) = \Phi'(2\pi) \Rightarrow i\mu(C - D) = i\mu(Ce^{i\mu \cdot 2\pi} - De^{-i\mu \cdot 2\pi})$$

$$\Rightarrow C(e^{i\mu \cdot 2\pi} - 1) = 0, \quad D(e^{-i\mu \cdot 2\pi} - 1) = 0 \Leftrightarrow D(e^{i\mu \cdot 2\pi} - 1) = 0$$

$$e^{i\mu \cdot 2\pi} = 1 \Rightarrow \mu = m \in \mathbb{N}^+$$

$$\lambda_m = m^2, \quad \Phi_m(\phi) = \{\cos m\phi, \sin m\phi\} \quad \text{简并}$$

$$\lambda_m = m^2, \quad \Phi_m(\phi) = \{\cos m\phi, \sin m\phi\}, \quad m \in \mathbb{N}^+ \quad \text{在 } [0, 2\pi] \text{ 正交完备}$$

$$\rho^2 R_m''(\rho) + \rho R_m'(\rho) - m^2 R_m(\rho) = 0 \quad \text{欧拉方程}$$

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dR_m}{d\rho} \right) - m^2 R_m = 0$$

$$\frac{d}{d \ln \rho} \frac{dR_m}{d \ln \rho} - m^2 R_m = 0$$

$$t = \ln \rho \Rightarrow \frac{d^2 R_m}{dt^2} - m^2 R_m = 0$$

$$m=0: \quad R_0 = \{1, t\} = \{1, \ln \rho\}$$

$$m \neq 0: \quad R_m = \{e^{mt}, e^{-mt}\} = \{\rho^m, \rho^{-m}\}$$

一般解: (方程、 ϕ 的周期性 BC, 不考虑 ρ 的 BC)

$$u(\rho, \phi) = A_0 + B_0 \ln \rho + \sum_{m=1}^{\infty} \rho^m (A_m \cos m\phi + B_m \sin m\phi) \\ + \sum_{m=1}^{\infty} \rho^{-m} (C_m \cos m\phi + D_m \sin m\phi)$$

$$\{1, \ln \rho\} + \sum_{m=1}^{\infty} \{\rho^m, \rho^{-m}\} \cdot \{\cos m\phi, \sin m\phi\}$$

$\ln \rho$

$$\Delta_2 u = 0 \Rightarrow u \sim \ln r \quad \frac{1}{r}$$

$$\Delta_3 u = 0 \Rightarrow u \sim \frac{1}{r} \quad \frac{1}{r^2}$$

$$\Delta_4 u = 0 \Rightarrow u \sim \frac{1}{r^2} \quad \frac{1}{r^3}$$

[空间是3维的]

§7.5.2 用边界条件确定系数

圆内 Laplace 方程定解问题

$$\nabla^2 u = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} = 0 \quad (\rho < a)$$

$$u|_{\rho=a} = F(\phi)$$

$$\text{补充 } u|_{\rho=0} < \infty$$

自然边界条件, $\rho=0$ 无电荷

$$\hookrightarrow B_0 = 0, C_m = D_m = 0$$

$$u(\rho, \phi) = A_0 + \sum_{m=1}^{\infty} \left(\frac{\rho}{a} \right)^m (A_m \cos m\phi + B_m \sin m\phi)$$

$$A_0 + \sum_{m=1}^{\infty} (A_m \cos m\phi + B_m \sin m\phi) = F(\phi)$$

$$\vec{A} = \sum \vec{A}_i \cdot \vec{e}_i$$

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} F(\phi) d\phi \quad (1, 1) = 2\pi$$

$$A_i = \vec{A} \cdot \vec{e}_i$$

$$A_m = \frac{1}{\pi} \int_0^{2\pi} F(\phi) \cos m\phi d\phi, \quad B_m = \frac{1}{\pi} \int_0^{2\pi} F(\phi) \sin m\phi d\phi$$

$$\text{例: } F(\phi) = 1 + \cos \phi + \sin 3\phi$$

Cauchy 积分公式

(19)

圆外 Laplace 方程定解问题

$$\nabla^2 u = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} = 0 \quad (\rho > a)$$

$$u|_{\rho=a} = G(\phi)$$

$$\text{补充 } u|_{\rho \rightarrow \infty} < \infty$$

自然 BC, ∞ 无电荷

$$\hookrightarrow B_0 = 0, A_m = B_m = 0$$

$$u(\rho, \phi) = A_0 + \sum_{m=1}^{\infty} \left(\frac{\rho}{a} \right)^{-m} (C_m \cos m\phi + D_m \sin m\phi)$$

$$A_0 + \sum_{m=1}^{\infty} (C_m \cos m\phi + D_m \sin m\phi) = G(\phi) \Rightarrow A_0, C_m, D_m$$

· 区域 $a < \rho < b$

· 边界条件对 x 轴有反射对称性: $u(\rho, -\phi) = u(\rho, \phi)$

$$u(\rho, \phi) = A_0 + B_0 \ln \rho + \sum_{m=1}^{\infty} (A_m \rho^m + C_m \rho^{-m}) \cos m\phi$$

§7.5.3 均匀电场中的导体圆柱

无限长圆柱导体, 半径为 a , 不带电, 置于均匀外电场 $\mathbf{E}_0$ 中, $\mathbf{E}_0$ 的方向垂直于柱轴. 求空间各处的电势分布.

这是一个综合应用题, 通过此类问题的求解, 读者可以逐步培养怎样从物理到数学, 再由数学到物理的分析解决问题的能力. 为了清楚起见, 下面将求解过程分为几步.

第一步 物理分析. 由于静电场中的导体为等势体, 故柱内和柱面的电势为常数, 可取为零. 关键是求解柱外的电势分布. 显然, 任何垂直于柱轴的平面上的物理状况都是相同的, 只需要选取其中一个平面作为代表来研究. 故本题是二维稳定场问题.

注 由于存在无限长圆柱, 选无穷远为电势零点一般是不合适的. 在本题中, 不同方向的无穷远点显然具有不同的电势, 所以更不能取无穷远点为电势零点.

第二步 建立坐标系. 取柱轴为 z 轴, 任取柱轴上一点为原点, 又取电场 $\mathbf{E}_0$ 的方向为 x 轴的方向. 如前所述, 所有的条件不随 z 变化, 任何垂直于 z 轴的平面上的电势分布都是相同的, 故只需计算 xy 平面上的电势. 鉴于边界的形状, 只能采用极坐标系.

第三步 建立定解问题. 由于柱外没有电荷, 故电势 $u(\rho, \phi)$ 满足 Laplace 方程. 本题有两个边界, 即柱面 (或圆周) 和无穷远处. 柱面的电势已取为零. 至于无穷远处, 原均匀电场应基本不受影响. 由此可列出以下定解问题:

$$\nabla^2 u = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} = 0, \quad (\rho > a) \quad (130a)$$

$$u|_{\rho=a} = 0, \quad (130b)$$

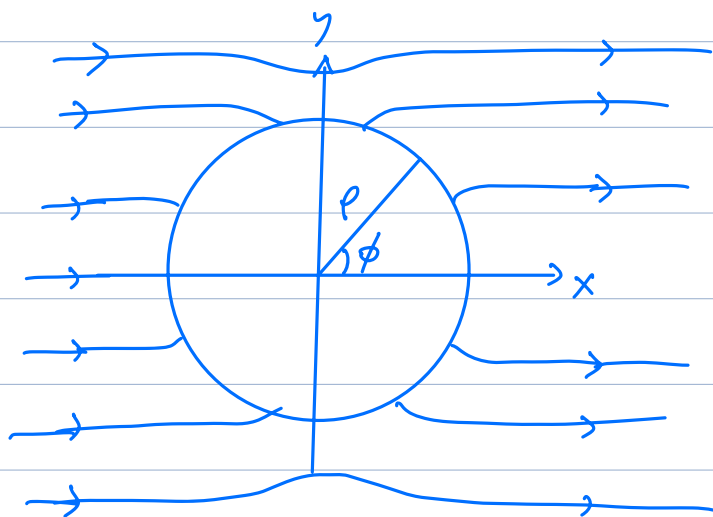
$$u|_{\rho \rightarrow \infty} = -E_0 \rho \cos \phi + u_0. \quad \text{渐近行为} \quad (130c)$$

其中 u_0 为常数, 需要通过求解来确定.

$$\rho \rightarrow \infty \quad \vec{E} = E_0 \vec{e}_x$$

$$\frac{\partial u}{\partial x} = -E_0, \quad \frac{\partial u}{\partial y} = 0 \quad \Rightarrow \quad u = -E_0 x + u_0 = -E_0 \rho \cos \phi + u_0.$$

与电势零点的选择有关



第四步 求解定解问题. 由前面求得的一般解, 考虑到本题具有对 x 轴的反射对称性, 可知本题的一般解为式 (129). 代入无穷远处的边界条件 (130c), 则得

$$A_0 + B_0 \ln \rho + \sum_{m=1}^{\infty} A_m \rho^m \cos m\phi = u_0 - E_0 \rho \cos \phi.$$

比较两边可得

$$A_0 = u_0, \quad B_0 = 0, \quad A_1 = -E_0, \quad A_m = 0 \quad (m > 1). \quad (131)$$

于是得到

$$u(\rho, \phi) = u_0 - E_0 \rho \cos \phi + \sum_{m=1}^{\infty} C_m \rho^{-m} \cos m\phi. \quad (132)$$

再代入柱面的边界条件 (130b), 可得

$$u_0 - E_0 a \cos \phi + \sum_{m=1}^{\infty} C_m a^{-m} \cos m\phi = 0.$$

比较两边可得

$$u_0 = 0, \quad C_1 = E_0 a^2, \quad C_m = 0 \quad (m > 1). \quad (133)$$

最后得到

$$u(\rho, \phi) = -E_0 \rho \cos \phi + \frac{E_0 a^2}{\rho} \cos \phi. \quad (134)$$

第五步 结果分析. 以上结果中, 第一项显然是原均匀电场的电势, 而第二项则是由柱面上的感应面电荷产生的. 不难求得柱面上的电荷面密度为

$$\sigma(\phi) = -\epsilon_0 \left. \frac{\partial u}{\partial n} \right|_{\rho=a} = -\epsilon_0 \left. \frac{\partial u}{\partial \rho} \right|_{\rho=a} = 2\epsilon_0 E_0 \cos \phi. \quad (135)$$

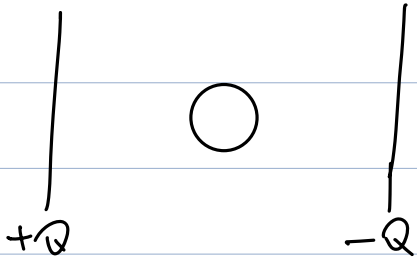
显然, 柱面上 $\phi \in (-\pi/2, \pi/2)$ 部分带正电, $\phi \in (\pi/2, 3\pi/2)$ 部分带负电, 而每单位长度的总带电量为 $Q = \int_0^{2\pi} \sigma(\phi) a d\phi = 2a\epsilon_0 E_0 \int_0^{2\pi} \cos \phi d\phi = 0$, 一切如所期望.

$$E_r = -\frac{\partial u}{\partial \rho} = E_0 \left(\frac{a^2}{\rho^2} + 1 \right) \cos \phi$$

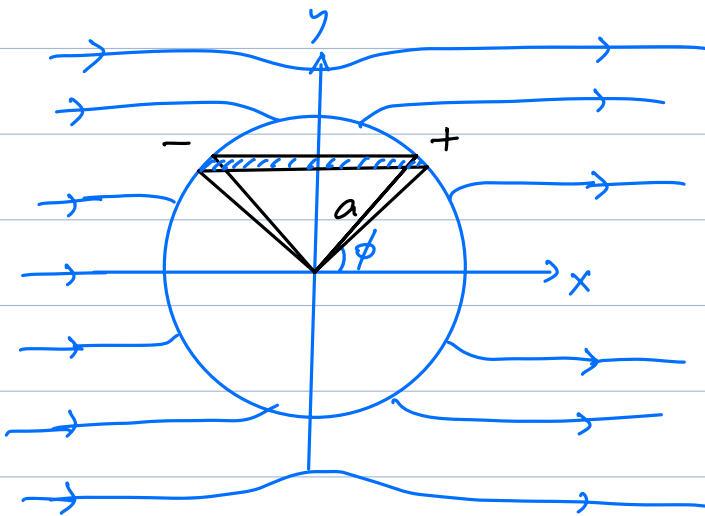
$$E_\phi = -\frac{1}{\rho} \frac{\partial u}{\partial \phi} = E_0 \left(\frac{a^2}{\rho^2} - 1 \right) \sin \phi$$

$$\rho = a: \quad E_r = 2E_0 \cos \phi, \quad E_\phi = 0$$

注: ①



②



$$2a \cos \phi$$

$$\sigma(\phi) a d\phi \cdot 2a \cos \phi = 4a^2 \epsilon_0 E_0 \cos^2 \phi d\phi$$

$$P = \int_{-\pi/2}^{\pi/2} 4a^2 \epsilon_0 E_0 \cos^2 \phi d\phi = 2\pi a^2 \epsilon_0 E_0$$

$$\text{2维电偶极子 } \vec{p} \text{ 产生的电势: } \frac{E_0 a^2}{\rho} \cos \phi$$

§7.5.4 Poisson 方程的处理

圆内定解问题

$$\begin{cases} \nabla^2 u = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} = f(\rho, \phi) & (\rho < a) \\ u|_{\rho=a} = F(\phi) \end{cases}$$

$$u(\rho, \phi) = v(\rho, \phi) + u_0(\rho, \phi)$$

$$\downarrow \text{满足 } \nabla^2 u_0(\rho, \phi) = f(\rho, \phi)$$

$$\begin{cases} \nabla^2 v = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial v}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 v}{\partial \phi^2} = 0 & (\rho < a) \\ v|_{\rho=a} = G(\phi) \equiv F(\phi) - u_0(\rho, \phi) \end{cases}$$

$$\text{例: } f(\rho, \phi) = 1$$

$$u_0 = \frac{x^2}{2} \text{ 或 } \frac{y^2}{2}$$

$$u_0 = \frac{1}{4}(x^2 + y^2) = \frac{1}{4}\rho^2$$

$$f(\rho, \phi) = a + b(x^2 - y^2)$$

$$u_0 = \frac{a}{4}(x^2 + y^2) + \frac{b}{12}(x^4 - y^4)$$

$$= \frac{a}{4}\rho^2 + \frac{b}{12}(x^2 + y^2)(x^2 - y^2)$$

$$= \frac{a}{4}\rho^2 + \frac{b}{12}\rho^2 \cos 2\phi$$

本征函数展开法

$$u(\rho, \phi) = A_0(\rho) + \sum_{m=1}^{\infty} (A_m(\rho) \cos m\phi + B_m(\rho) \sin m\phi)$$

$$f(\rho, \phi) = a_0(\rho) + \sum_{m=1}^{\infty} (a_m(\rho) \cos m\phi + b_m(\rho) \sin m\phi)$$

$$F(\phi) = \alpha_0 + \sum_{m=1}^{\infty} (\alpha_m(\rho) \cos m\phi + \beta_m(\rho) \sin m\phi)$$

$$(\rho A_0')' = \rho a_0, \quad \rho^2 A_m'' + \rho A_m' - m^2 A_m = \rho^2 a_m$$

$$\rho^2 B_m'' + \rho B_m' - m^2 B_m = \rho^2 b_m$$

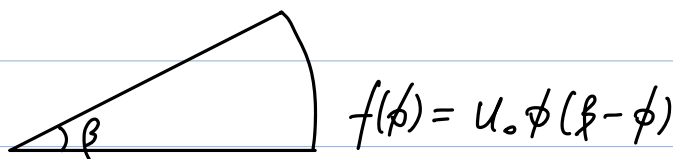
$$A_0(a) = \alpha_0, \quad A_m(a) = \alpha_m, \quad B_m(a) = \beta_m$$

$$\rho=0 \text{ 处 } \phi \text{ 没有定义 } \Rightarrow A_m(0) = 0, \quad B_m(0) = 0, \quad m \in \mathbb{N}^+$$

$$|A_0(0)| < \infty$$

自然边界条件

习题



$$\Phi(0) = 0, \quad \Phi(\beta) = 0$$

§7.6 * 解的唯一性与稳定性

§1 Fourier 变换

§1.1 复数形式的 Fourier 级数

$f(x)$ 以 $2l$ 为周期: $f(x+2l) = f(x)$

$$\left\{ \cos \frac{n\pi x}{l}, \sin \frac{n\pi x}{l} \right\}_{n=0}^{\infty} \Leftrightarrow \left\{ e^{i \frac{n\pi x}{l}} \right\}_{n=-\infty}^{\infty}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

正交

$[-l, l]$

系数

$$a_0 = \frac{1}{2l} \int_{-l}^l f(\xi) d\xi$$

$$a_n = \frac{1}{l} \int_{-l}^l f(\xi) \cos \frac{n\pi \xi}{l} d\xi$$

$$b_n = \frac{1}{l} \int_{-l}^l f(\xi) \sin \frac{n\pi \xi}{l} d\xi$$

Fourier series, coefficient

不重不漏 $\leftrightarrow$ 正交完备

完备性: 均方收敛性

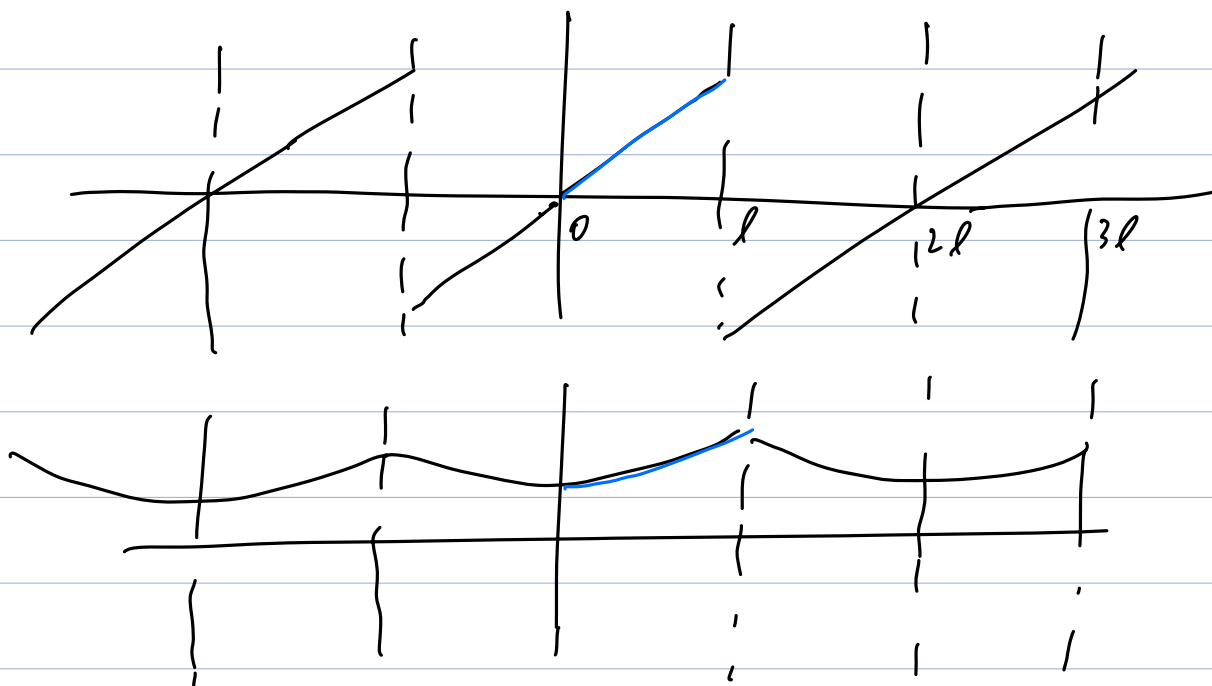
收敛 Dirichlet thm

奇函数: 傅里叶正弦变换

偶 ... : ... 余 ...

$$\varphi(x) = \frac{2}{l} \int_0^l \varphi_n \sin \frac{n\pi x}{l} dx$$

$$\varphi(x) = \frac{2}{l} \int_0^l \varphi_n \cos \frac{n\pi x}{l} dx$$



$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i \frac{n\pi x}{l}}$$

$$\begin{aligned} (e^{i \frac{m\pi x}{l}}, e^{i \frac{n\pi x}{l}}) &\equiv \int_{-l}^l (e^{i \frac{m\pi x}{l}})^* e^{i \frac{n\pi x}{l}} dx = \int_{-l}^l e^{i \frac{(n-m)\pi x}{l}} dx \\ &= 2l \delta_{mn} \end{aligned}$$

$$f_n = \frac{1}{2l} \int_{-l}^l f(\xi) e^{-i \frac{n\pi \xi}{l}} d\xi, \quad n \in \mathbb{Z}$$

内积定义中出现的复共轭，实函数内积的推广。

$$|A|^2 = \vec{A} \cdot \vec{A}$$

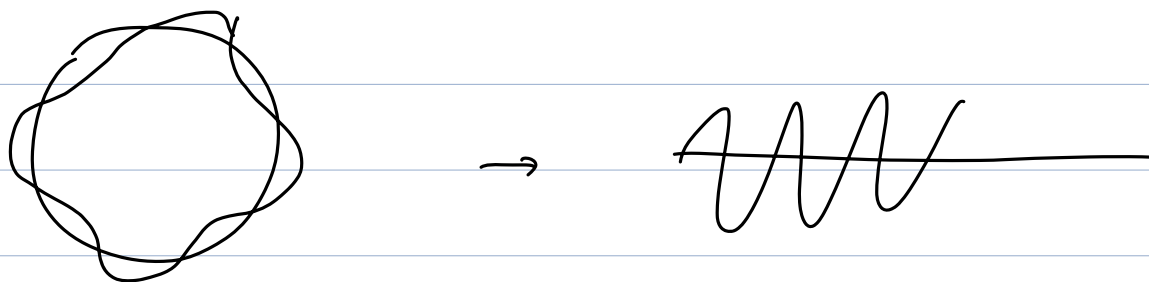
$$|z|^2 = z^* z$$

$$\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n), \quad \vec{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$$

$$(\vec{\alpha}, \vec{\beta}) = \sum_{i=1}^n \alpha_i^* \beta_i$$

§ 1.2 Fourier 变换

$l \rightarrow \infty$ 极限



$$\left\{ e^{i \frac{n\pi x}{l}} \right\}$$

$$n \in \mathbb{Z}$$

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i \frac{n\pi x}{l}}$$

$$\{ e^{ikx} \}$$

$$k \in \mathbb{R}$$

$$f(x) = \sum_{k \in \mathbb{R}} F_k e^{ikx}$$

$$= \# \int_{-\infty}^{\infty} F(k) e^{ikx} dk$$

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i \frac{n\pi x}{l}}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2l} \int_{-l}^l f(\xi) e^{-i \frac{n\pi \xi}{l}} d\xi \right] e^{i \frac{n\pi x}{l}}$$

$$= \sum_{k_n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-l}^l f(\xi) e^{-ik_n \xi} d\xi \right] e^{ik_n x} \cdot \frac{1}{l} \Delta k_n$$

$$l \rightarrow \infty = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{ik\xi} d\xi \right] e^{ikx} dk$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk$$

对 比:

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i \frac{n\pi x}{L}}$$

$$f_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-i \frac{n\pi x}{L}} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

对称性 $\Rightarrow$ 守恒律
(守恒量)

time translation sym. $\Rightarrow$ conservation of energy

space - - - - $\Rightarrow$ - - - - momentum

rotational sym. $\Rightarrow$ - - - - angular momentum

$U(1)$ $\Rightarrow$ - - - - electric charge

$$e^{i\omega t}, e^{ikx}$$

$$f(x) \leftrightarrow F(k)$$

$$\text{或 } F(k) = \mathcal{F}[f(x)], \quad f(x) = \mathcal{F}^{-1}[F(k)]$$

§1.3 = §§§ § = §§§ Fourier 变换

$$F(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} f(\vec{r}) e^{-i\vec{k} \cdot \vec{r}} d\vec{r}$$

$$f(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} F(\vec{k}) e^{i\vec{k} \cdot \vec{r}} d\vec{r}$$

§1.4 Fourier 变换的性质

$$f(x) \leftrightarrow F(k)$$

$$(1) \quad a_1 f_1(x) + a_2 f_2(x) \leftrightarrow a_1 F_1(k) + a_2 F_2(k)$$

$$(2) \quad \text{微分定理: 如果 } f(\pm\infty) = f'(\pm\infty) = \dots = f^{(n-1)}(\pm\infty) = 0, \\ \text{则 } f^{(n)}(x) \leftrightarrow (ik)^n F(k)$$

Pf

$$\begin{aligned} f'(x) &\leftrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} f(x) e^{ikx} \Big|_{-\infty}^{\infty} + \frac{ik}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\ &= ik F(k) \end{aligned}$$

$$(3) \quad \text{卷积定理: } f_1(x) * f_2(x) \leftrightarrow F_1(k) F_2(k)$$

$$f_1(x) * f_2(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(\xi) f_2(x-\xi) d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x-\xi) f_2(\xi) d\xi$$

用于由像函数求原函数.

$$(4) \quad \text{延迟定理} \quad f(x-\xi) \leftrightarrow e^{-ik\xi} F(k), \quad \xi \in \mathbb{R}$$

(5) 积分定理

$$\text{如果 } \int_{-\infty}^{\infty} f(x) dx = 0, \text{ 则 } g(x) \equiv \int_{-\infty}^x f(\xi) d\xi \leftrightarrow \frac{F(k)}{ik}$$

(6) 相似定理

$$f(ax) = \frac{1}{|a|} F\left(\frac{k}{a}\right), \quad a \neq 0$$

§8.2 无界弦的自由振动

无穷长的弦：在较短时间内，边界对所关心部分尚未发生相互作用

§8.2.1 Fourier 变换法

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (-\infty < x < +\infty, t > 0)$$

$$u|_{t=0} = \varphi(x) \quad (-\infty < x < +\infty)$$

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(x) \quad (-\infty < x < +\infty)$$

$$u|_{x=\pm\infty} = 0, \quad \left. \frac{\partial u}{\partial x} \right|_{x=\pm\infty} = 0 \quad \leftarrow \text{Fourier 变换的条件}$$

$t \in (0, \infty)$, $x \in (-\infty, \infty)$ ：应该对 x 作 Fourier 变换

$$u(x, t) \leftrightarrow U(k, t), \quad \varphi(x) \leftrightarrow \Phi(k), \quad \psi(x) \leftrightarrow \Psi(k)$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} e^{-ikx} dx - \frac{a^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-ikx} dx = 0$$

$$u(x, t) \sim e^{-ikx} U(k, t)$$

有界弦： $u \sim \delta(x) T(t)$

$$\begin{cases} \frac{d^2 U}{dt^2} + k^2 a^2 U = 0 \\ U|_{t=0} = \Phi(k), \quad \left. \frac{\partial U}{\partial t} \right|_{t=0} = \Psi(k) \end{cases}$$

$$\Rightarrow U(k, t) = \Phi(k) \cos kat + \Psi(k) \frac{\sin kat}{ka}$$

$$\Rightarrow u(x, t) = \mathcal{F}^{-1}[U(k, t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(k, t) e^{ikx} dk$$

$$F^{-1}[\Phi(k) \cos kat] = \frac{1}{2} \cdot \frac{1}{\omega a} \int_{-\infty}^{\infty} \Phi(k) (e^{ik(x-at)} + e^{ik(x+at)}) dk$$

$$= \frac{1}{2} [\varphi(x-at) + \varphi(x+at)]$$

$$F^{-1}\left[\Phi(k) \frac{\sin kat}{ka}\right] = F^{-1}\left[\int_0^t \Phi(k) \cos ka\tau d\tau\right] = \int_0^t F^{-1}[\Phi(k) \cos ka\tau] d\tau$$

$$= \frac{1}{2} \int_0^t [\varphi(x-a\tau) + \varphi(x+a\tau)] d\tau$$

$$= \frac{1}{2a} \int_{x-at}^x \psi(\xi) d\xi + \frac{1}{2a} \int_x^{x+at} \psi(\xi) d\xi = \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi$$

$$u(x,t) = \frac{1}{2} [\varphi(x-at) + \varphi(x+at)] + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi$$

达朗贝尔公式

若 $\psi(x) = 0$ $u(x,t) = \frac{1}{2} [\varphi(x-at) + \varphi(x+at)]$

$\downarrow$ 向右传播 $\downarrow$ 向左传播

有界法

$$u \sim \delta(x) T(t)$$

分离变量

$$u(x,t) = \sum_n T_n(t) \delta_n(x)$$

无界法

$$u \sim e^{ikx} U(k,t)$$

$$u(x,t) = \int_{-\infty}^{\infty} U(k,t) e^{ikx} dk$$

思：用延迟定理、线性定理 $\rightarrow F^{-1}[\Phi(k) \cos kat]$

-----、积分定理 $\rightarrow F^{-1}\left[\Phi(k) \frac{\sin kat}{ka}\right]$

§8.2 行波法简介

1+1 维

$$\left(\frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial x^2}\right) u = 0$$

$$\left(\frac{\partial}{\partial t} + a \frac{\partial}{\partial x}\right) \left(\frac{\partial}{\partial t} - a \frac{\partial}{\partial x}\right) u$$

$$\xi = x - at, \quad \eta = x + at$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t} \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

$$\frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \xi} \right) = 0 \Rightarrow \frac{\partial u}{\partial \xi} = f_0(\xi) \Rightarrow u = \int f_0(\xi) d\xi + g(\eta) \\ = f(\xi) + g(\eta)$$

通解: $u(x, t) = f(x - at) + g(x + at)$

$$f(x) + g(x) = \varphi(x)$$

$$-af'(x) + ag'(x) = \psi(x)$$

思: 1. 方程没有解出具体函数

2. $x \rightarrow x, t \rightarrow \gamma, a \rightarrow i$

$$(\partial_t^2 - a^2 \partial_x^2) u = 0$$

通解 $u(x, t) = f(x - at) + g(x + at)$

$$a \rightarrow i, \quad t \rightarrow \gamma$$

$$\partial \bar{\partial} u = 0$$

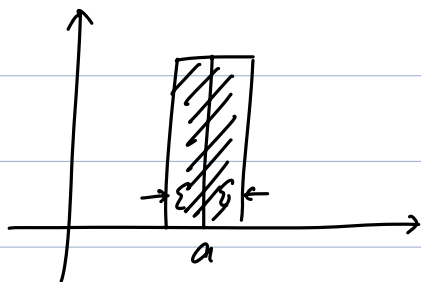
$$u(x, \gamma) = f(z) + g(\bar{z})$$

§8.3 δ 函数

§8.3.1 定义一

x 轴上 $x=a$ 处有单位质量的质点, 质量线密度:

$$\delta_\varepsilon(x-a) = \begin{cases} \frac{1}{2\varepsilon}, & |x-a| < \varepsilon \\ 0, & |x-a| > \varepsilon \end{cases}$$



$$\delta(x-a) = \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon(x-a)$$

定义一:
$$\delta(x-a) = \begin{cases} \infty, & x=a \\ 0, & x \neq a \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x-a) dx = 1$$

δ 函数是广义函数

把 δ 函数当作某种经典函数序列的极限

δ 函数的量纲是其宗量量纲的倒数

$e^x \rightarrow$ 无量纲

$\delta(x)$ 有量纲

§ 8.3.2 定义 =

对于任何连续函数 $f(x)$

$$\int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a)$$

积分中值定理:

$$\int_a^b f(x) g(x) dx = f(\xi) \int_a^b g(x) dx$$

Pf

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \delta(x-a) dx &= \int_{a-\varepsilon}^{a+\varepsilon} f(x) \delta(x-a) dx = f(a+\theta\varepsilon) \int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a) dx \\ &= f(a+\varepsilon\theta) \xrightarrow{\varepsilon \rightarrow 0} f(a) \end{aligned}$$

3 维: $\int f(\vec{r}) \delta(\vec{r}-\vec{a}) d\vec{r} = f(\vec{a})$

§ 8.3.3 δ 函数的基本性质

1. 若 $f(x)$ 为连续函数, 则 $f(x) \delta(x-a) = f(a) \delta(x-a)$
2. $x \delta(x) = 0$
3. 偶函数: $\delta(-x) = \delta(x)$
4. $\delta(ax) = \frac{1}{|a|} \delta(x)$

思: ? $\xrightarrow{\text{求导}}$? $\xrightarrow{\text{求导}}$ $\delta(x)$

$\theta(x)$

$$\frac{d|x|}{dx} = \text{sgn}(x) = 2\theta(x) - 1$$

§8.3.4 δ 函数的几种表达式

1. $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$

2. $\delta(x) = \lim_{k \rightarrow \infty} \frac{\sin Kx}{\pi x}$

3. $\delta(x) = \lim_{\sigma \rightarrow 0} \rho_{\sigma}(x)$, $\rho_{\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$ Gauss 分布

4. $\delta(x) = \lim_{b \rightarrow 0} \rho_b(x)$, $\rho_b(x) = \frac{b}{\pi(x^2 + b^2)}$ Lorentz 分布

5. $\delta(x) = \theta'(x)$, $\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$ Heaviside 阶跃函数

1. $\delta(x)$ 的 Fourier 变换为

$$\Delta(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}}$$

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Delta(k) e^{ikx} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$$

$$\Delta_{\varepsilon}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta_{\varepsilon}(x) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2\varepsilon} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \frac{\sin k\varepsilon}{k\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{\sqrt{2\pi}}$$

$$\delta_{\varepsilon}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Delta_{\varepsilon}(k) e^{ikx} dk \quad \varepsilon \rightarrow 0$$

2. $\delta(x) = \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \Delta_{\varepsilon}(k) e^{ikx} dk = \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \rightarrow 0} \lim_{K \rightarrow \infty} \int_{-K}^K \Delta_{\varepsilon}(k) e^{ikx} dk$

$$= \frac{1}{\sqrt{2\pi}} \lim_{K \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \int_{-K}^K \Delta_{\varepsilon}(k) e^{ikx} dk$$

$$= \frac{1}{2\pi} \lim_{K \rightarrow \infty} \int_{-K}^K e^{ikx} dk = \frac{1}{\pi} \lim_{K \rightarrow \infty} \int_{-K}^K \cos kx dk = \lim_{K \rightarrow \infty} \frac{\sin Kx}{\pi x}$$

$$3. \quad x \neq 0, \quad \lim_{\sigma \rightarrow 0} \rho_{\sigma}(x) = 0$$

$$x = 0, \quad \lim_{\sigma \rightarrow 0} \rho_{\sigma}(x) = \infty$$

4. 任何概率分布取其宽度趋于零 $\rightarrow \delta$ 函数

$$\begin{aligned} 5. \quad \int_{-\infty}^{\infty} f(x) \theta'(x) dx &= f(x) \theta(x) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f'(x) \theta(x) dx \\ &= f(+\infty) - \int_0^{+\infty} f'(x) dx \\ &= f(+\infty) - f(x) \Big|_0^{+\infty} = f(0) \end{aligned}$$

§8.3.5 * δ 函数的导数

$$\int_{-\infty}^{\infty} f(x) \delta'(x-a) dx = -f'(a)$$

§8.3.6 * 复合 δ 函数

$$\delta[\varphi(x)] = \sum_k \frac{1}{|\varphi'(x_k)|} \delta(x-x_k)$$

坐标空间、动量空间

原子、原子核 尺度 $\leftrightarrow$ 能量 对撞机

§8.4 无界杆的热传导问题

$$\begin{cases} \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f & (-\infty < x < \infty, t > 0) \\ u|_{t=0} = \varphi(x) & (-\infty < x < \infty) \end{cases}$$

§8.4.1 无源问题

$$\begin{cases} \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 & (-\infty < x < \infty, t > 0) \\ u|_{t=0} = \varphi(x) & (-\infty < x < \infty) \end{cases}$$

对 $x \in \mathbb{R}$ Fourier 变换

$$u(x, t) \leftrightarrow U(k, t), \quad \varphi(x) \leftrightarrow \Phi(k)$$

$$\begin{cases} \frac{dU}{dt} + k^2 a^2 U = 0 \\ U|_{t=0} = \Phi(k) \end{cases}$$

$$\Rightarrow U(k, t) = \underbrace{\Phi(k)}_{\varphi(x) * \psi(x)} e^{-k^2 a^2 t}$$

$$\begin{aligned} \psi(x) &= \mathcal{F}^{-1}\{e^{-k^2 a^2 t}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2 a^2 t} e^{ikx} dk = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-k^2 a^2 t} \cos kx dx \\ &= \frac{1}{a\sqrt{2t}} e^{-\frac{x^2}{4a^2 t}} \end{aligned}$$

$$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi$$

$$\text{特例: } \varphi(x) = \delta(x-x_0), \quad u(x, t) = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-x_0)^2}{4a^2 t}}$$

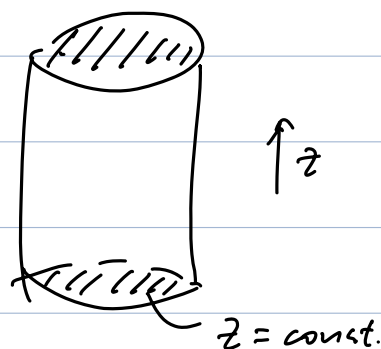
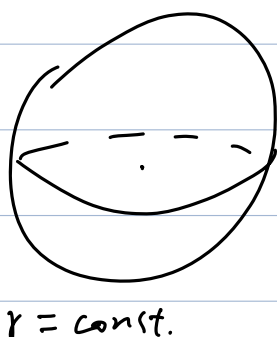
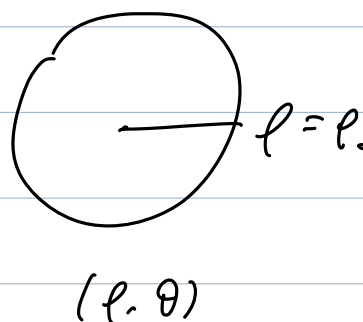
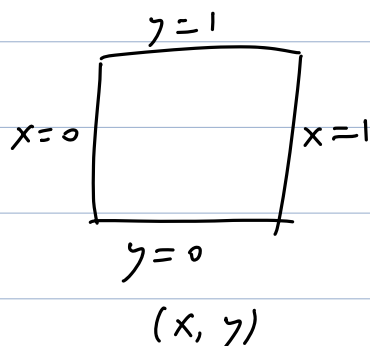
总能量守恒; 只要 $t > 0$, 杆上各处的温度均不为零; t 较大之后合理.

Chap. 9 正交曲线坐标系中的分离变量

§9.1 正交曲线坐标系中的微分算符

(x, y, z) 坐标线 $x = \text{const.}$
 y
 z

坐标线和边界重合.



(r, θ, φ)

(ρ, θ, z)

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = \sum_{i,j=1}^3 \frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j} dq_i dq_j.$$

$$\text{正交: } (ds)^2 = \sum_{i=1}^3 \left(\frac{\partial \mathbf{r}}{\partial q_i} \right)^2 (dq_i)^2 \equiv \sum_{i=1}^3 \underset{\substack{\uparrow \\ \text{度规系数}}}{h_i} (dq_i)^2,$$

$$ds^2 = \sum_{\mu, \nu} g_{\mu\nu} dx^\mu dx^\nu$$

$$\nabla^2 u = \sum_{\mu, \nu} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{g} g^{\mu\nu} \frac{\partial u}{\partial x^\nu} \right)$$

$$g = \det(g_{\mu\nu}), \quad g^{\mu\nu} = (g^{-1})_{\mu\nu}, \quad \text{i.e.} \quad g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$$

$$\begin{aligned}
 ds^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\
 &= 1 \cdot dx dx + 0 \cdot dx dy + 0 \cdot dx dz \\
 &\quad + 0 \cdot dy dx + 1 \cdot dy dy + 0 \cdot dy dz \\
 &\quad + 0 \cdot dz dx + 0 \cdot dz dy + 1 \cdot dz dz
 \end{aligned}$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases}$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad (h_r = 1, h_\theta = r, h_\varphi = r \sin \theta)$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2 \theta \end{pmatrix} \quad g^{\mu\nu}$$

$$g = \det(g_{\mu\nu}) = r^4 \sin^2 \theta$$

$$\nabla^2 u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{g} g^{\mu\nu} \frac{\partial u}{\partial x^\nu} \right)$$

$$\begin{aligned}
 &= \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(r^2 \sin \theta \frac{1}{r^2} \frac{\partial u}{\partial \theta} \right) \\
 &\quad + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \varphi} \left(r^2 \sin \theta \cdot \frac{1}{r^2 \sin \theta} \frac{\partial u}{\partial \varphi} \right)
 \end{aligned}$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial^2 u}{\partial \varphi^2}$$

$$\begin{cases} x = \rho \cos \varphi \\ y = \rho \sin \varphi \\ z \end{cases}$$

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

$$(h_\rho = 1, h_\varphi = \rho, h_z = 1)$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & \rho^2 & \\ & & 1 \end{pmatrix}$$

$$g = \det(g_{\mu\nu}) = \rho^2$$

$$\nabla^2 u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} (\sqrt{g} g^{\mu\nu} \frac{\partial u}{\partial x^\nu})$$

$$= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial}{\partial \varphi} \left(\rho \frac{1}{\rho^2} \frac{\partial u}{\partial \varphi} \right) + \frac{1}{\rho} \frac{\partial}{\partial z} \left(\rho \frac{\partial u}{\partial z} \right)$$

$$= \underbrace{\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial^2 u}{\partial \varphi^2}}_{\text{柱坐标 } (\rho, \varphi)} + \frac{\partial^2 u}{\partial z^2}$$

柱坐标 (ρ, φ)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

3维欧氏空间 (x, y, z)

(r, θ, φ)

(ρ, φ, z)

原始人 $\rightarrow$ 穿衣人

弯曲空间、弯曲时空

§ 9.2 球坐标系中的分离变量

§ 9.2.1 Helmholtz 方程

亥姆霍兹方程

$$\partial_t^2 u - a^2 \nabla^2 u = 0$$

$$u = v(\vec{r}) T(t)$$

$$v T'' - a^2 (\nabla^2 v) \cdot T = 0$$

$$\frac{\nabla^2 v}{v} = \frac{T''}{a^2 T} = -k^2$$

$$\nabla^2 v + k^2 v = 0$$

$$T'' + k^2 a^2 T = 0$$

$$(T'' - k^2 a^2 T = 0 \Rightarrow T \sim e^{\pm i k a t} \text{ 不是波动})$$

$$\partial_t^2 u - a^2 \nabla^2 u = 0$$

$$u = v(\vec{r}) T(t)$$

$$\nabla^2 v + k^2 v = 0$$

$$T' + k^2 a^2 T = 0$$

$$(T' - k^2 a^2 T = 0 \Rightarrow T \sim e^{i k^2 a^2 t} \text{ 不合理})$$

$$\nabla^2 u = 0 \quad \text{Laplace eq.}$$

$$(\nabla^2 + k^2) u = 0 \quad \text{Helmholtz eq.}$$

$$\nabla^2 u = f \quad \text{Poisson eq.}$$

§9.2.2 Helmholtz 方程在球坐标系中的分离变量

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial^2 u}{\partial \varphi^2} + k^2 u = 0$$

$$u = R(r) Y(r, \theta)$$

$$\begin{aligned} & \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 R'(r) \cdot Y \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \cdot R \frac{\partial Y}{\partial \theta} \right) \\ & + \frac{1}{r^2 \sin \theta} R \cdot \frac{\partial^2 Y}{\partial \varphi^2} + k^2 R Y = 0 \end{aligned}$$

$$\times \frac{r^2}{R Y} \Rightarrow$$

$$\frac{1}{R} (r^2 R')' + k^2 r^2 = - \frac{1}{Y} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \varphi^2} \right] = \lambda$$

$$r^2 R'' + 2r R' + (k^2 r^2 - \lambda) R = 0$$

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \varphi^2} + \lambda Y = 0$$

$$Y(\theta, \varphi) = H(\theta) \Phi(\varphi) \quad (\Theta(\theta))$$

$$\times \frac{\sin^2 \theta}{H \Phi} \Rightarrow$$

$$\frac{1}{H} \left[\sin \theta \frac{d}{d\theta} \left(\sin \theta \frac{dH}{d\theta} \right) + \lambda \sin^2 \theta H \right] = - \frac{\Phi''}{\Phi} = \mu$$

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{dH}{d\theta} \right) + \left(\lambda - \frac{\mu}{\sin^2 \theta} \right) H = 0$$

$$\begin{cases} \Phi'' + \mu \Phi = 0 \\ \Phi(\varphi + 2\pi) = \Phi(\varphi) \end{cases}$$

本征值问题是

$$\mu = m^2, \quad \Phi(\varphi) = \{ e^{im\varphi}, e^{-im\varphi} \}, \quad m = 0, 1, 2, \dots$$

将 $\mu = m^2$ 代入, $x = \cos \theta, \quad H(\theta) = P(x)$

$$\begin{cases} \frac{d}{dx} \left[(1-x^2) \frac{dP}{dx} \right] + \left(\lambda - \frac{m^2}{1-x^2} \right) P = 0 \\ P(\pm 1) = 0 \quad (m \neq 0) \quad \text{或} \quad |P(\pm 1)| < \infty \quad (m = 0) \end{cases}$$

自然边界条件

连带勒让德方程 (Associated Legendre equation)

若问题的解与 φ 无关, $m = 0$ 轴对称

$$\begin{cases} \frac{d}{dx} \left[(1-x^2) \frac{dP}{dx} \right] + \lambda P = 0 \\ |P(\pm 1)| < \infty \end{cases} \quad \text{Legendre 方程}$$

习题: Schrödinger 方程

$$-\frac{\hbar^2}{2\mu} \nabla^2 u + V(r)u = Eu$$

§9.3 柱坐标系中的分离变量

对比球坐标

$$\nabla^2 u = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad (\rho < a)$$

$$(\alpha \partial_\rho u + \beta u) \big|_{\rho=a} = 0$$

也可在上下底加齐次BC
或(半)无限长

$$u(\rho, \varphi, z) = R(\rho) \Phi(\varphi) Z(z)$$

$$\frac{1}{\rho} \frac{d}{d\rho} (\rho R' \Phi Z) + \frac{1}{\rho^2} R \Phi'' Z + R \Phi Z'' = 0$$

$$\times \frac{\rho^2}{R \Phi Z}$$

$$\frac{\rho}{R} \frac{d}{d\rho} (\rho R') + \rho^2 \frac{Z''}{Z} = - \frac{\Phi''}{\Phi} = \mu$$

$$* \begin{cases} \Phi'' + \mu \Phi = 0 \\ \Phi(\varphi + 2\pi) = \Phi(\varphi) \end{cases}$$

$$\Rightarrow \mu = m^2, \quad \Phi(\varphi) = \{ \cos m\varphi, \sin m\varphi \}, \quad m = 0, 1, \dots$$

$$\frac{1}{R\rho} \frac{d}{d\rho} (\rho R') - \frac{\mu}{\rho^2} = - \frac{Z''}{Z} = -\lambda$$

$$* \quad Z'' - \lambda Z = 0$$

$$* \quad \frac{1}{\rho} \frac{d}{d\rho} (\rho R') + \left(\lambda - \frac{\mu}{\rho^2} \right) R = 0$$

边界条件

$$\alpha R'(a) + \beta R(a) = 0$$

$$R(0) < \infty$$

本征值 $\lambda \geq 0$ (S-L 本征值问题一般结论)

$$\lambda > 0: \quad x = \sqrt{\lambda} \rho, \quad R(\rho) = z(x)$$

$$z'' + \frac{1}{x} z' + \left(1 - \frac{m^2}{x^2}\right) z = 0$$

m 阶 Bessel 方程

$$\lambda = 0: \quad \rho^2 R'' + \rho R' - m^2 R = 0$$

$$R(\rho) = \begin{cases} \rho^m, \rho^{-m} & (m \neq 0) \\ \{1, \ln \rho\} & (m = 0) \end{cases}$$

$$BC: \quad \alpha R'(a) + \beta R(a) = 0$$

$$\text{当 } m \neq 0, \quad R(\rho) \sim \rho^m \Rightarrow m\alpha + \beta a = 0 \quad \text{不满足}$$

$$\text{当 } m = 0, \quad R(\rho) \sim 1 \Rightarrow \beta = 0 \quad \text{第二类齐次 BC}$$

$$Z(z) = \{1, z\}$$

Chap. 10 = 阶线性常微分方程的级数解法和一般本征值问题

一阶线性 ODE 的解可以用积分表出

= 极少 - - - - $\rightarrow$ 级数法

§1 常点邻域的级数解法

$$y'(x) + p(x)y(x) = Q(x)$$

$$y'' + p(x)y' + q(x)y = 0 \quad (10.1)$$

$\swarrow$
已知函数

x : 看作复数

$$IC: \quad y(x_0) = C_0, \quad y'(x_0) = C_1 \quad (10.2)$$

方程的常点和奇点

$p(x), q(x)$ 在 x_0 解析 $\rightarrow$ 方程的常点

极点, 本性奇点 $\rightarrow$ 奇点

Thm 如果 $p(x)$ 和 $q(x)$ 在圆 $|x - x_0| < R$ 内解析, 则在该圆内存在唯一的解析函数 $y(x)$ 满足方程 (10.1) 和初始条件 (10.2).

系数解析 $\Rightarrow$ 方程的解解析

$$p(x) = \sum_{k=0}^{\infty} p_k (x - x_0)^k, \quad q(x) = \sum_{k=0}^{\infty} q_k (x - x_0)^k, \quad y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

$$\text{代入方程得} \quad \sum_{k=0}^{\infty} (-p_k - q_k - a_k \dots) (x - x_0)^k = 0$$

所有系数为零 $\Rightarrow$ 解出 a_k

$$a_0 = C_0, \quad a_1 = C_1 \quad \Rightarrow \quad a_2, a_3, \dots$$

补充: (1) 已知一个解 $y_1(x)$ 可以将第二个解 $y_2(x)$ 用积分表达

$$y_2(x) = y_1(x) \int^x \frac{1}{y_1^2(u)} e^{-\int^u p(v) dv}$$

(2) 已知齐次方程的解 $y_1(x), y_2(x)$, 可将非齐次方程的一个特解用积分表达.

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x).$$

$$\Delta = y_2 y_1' - y_1 y_2'$$

$$Y(x) = y_2(x) \int^x \frac{f(u) y_1(u)}{\Delta(u)} du - y_1(x) \int^x \frac{f(u) y_2(u)}{\Delta(u)} du$$

§10.2 Legendre 方程及其本征值问题

§10.2.1 Legendre 方程的级数解

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$

$$(x = \cos \theta)$$

$$p(x) = -\frac{2x}{1-x^2}, \quad q(x) = \frac{\lambda}{1-x^2}$$

$$x = \pm 1, \infty \quad \text{奇点}$$

$$x=0 \quad \text{常点} \quad |x| < 1$$

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$\begin{aligned}
 y &= a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k \\
 y' &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (k+1)a_{k+1} x^k + \cdots \\
 y'' &= 2a_2 + 6a_3 x + \cdots + (k+2)(k+1)a_{k+2} x^k + \cdots = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k
 \end{aligned}$$

$$y'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1} \quad x y' = \sum_{k=0}^{\infty} k a_k x^k$$

$$y''(x) = \sum_{k'=0}^{\infty} k'(k'-1) a_{k'} x^{k'-2} = \sum_{k=-2}^{\infty} (k+2)(k+1) a_{k+2} x^k = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k$$

$\uparrow$
 $k = k' - 2$

$$x^2 y'' = \sum_{k=0}^{\infty} k(k-1) a_k x^k$$

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} - k(k+1) a_k + \lambda a_k] = 0$$

$$a_{k+2} = \frac{k(k+1) - \lambda}{(k+2)(k+1)} a_k, \quad k = 0, 1, 2, \dots$$

$$a_0 \rightarrow a_2 \rightarrow a_4 \rightarrow \cdots \rightarrow a_{2k}$$

$$a_1 \rightarrow a_3 \rightarrow a_5 \rightarrow \cdots \rightarrow a_{2k+1}$$

$$y(x) = a_0 y_0(x) + a_1 y_1(x)$$

$$y_0(x) = \sum_{k=0}^{\infty} C_{2k} x^{2k}, \quad y_1(x) = \sum_{k=0}^{\infty} C_{2k+1} x^{2k+1}$$

收敛半径

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+2}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{k(k+1) - \lambda}{(k+2)(k+1)} \right| = 1$$

$$l, \quad R = \frac{1}{l}$$

高斯判别法 $y_0(\pm 1) = \infty, \quad y_1(\pm 1) = \infty$

$$y_0(x) \sim \ln \frac{1}{1-x^2}$$

$$y_1(x) \sim \ln \frac{1+x}{1-x}$$

Parity

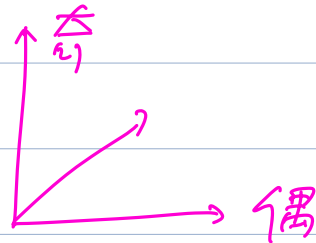
$$y(x) \Rightarrow y(-x)$$

对称性 (离散)

$$Py(x) := y(-x)$$

$$Py(x) = +y(x) \quad \text{偶函数}$$

$$Py(x) = -y(x) \quad \text{奇函数}$$



§10.2.2 Legendre 方程的本征值问题

自然边界条件: $|y(\pm 1)| < \infty$

$y_0(x)$, $y_1(x)$ 不满足这一条件, 唯一的出路是让它们中断为多项式.

$$\lambda = l(l+1), \quad l \in \mathbb{N}$$

$$* \quad \lambda = 2n(2n+1) \quad (n \in \mathbb{N}) \quad a_{2n+2} = a_{2n+4} = \dots = 0$$

$$y_0(x) \text{ 为 } 2n \text{ 次多项式}, \quad y_1(\pm 1) = \infty$$

$$* \quad \lambda = (2n+1)(2n+2) \quad (n \in \mathbb{N}) \quad a_{2n+3} = a_{2n+5} = \dots = 0$$

$$y_1(x) \text{ 为 } 2n+1 \text{ 次多项式}, \quad y_2(\pm 1) = \infty$$

适当选择 a_0 或 a_1 使 $a_{2l} = \frac{(2l)!}{2^l(l!)^2}, \quad l \in \mathbb{N}$

l 次 Legendre 多项式:

$$P_l(x) = \sum_{k=0}^{\lfloor l/2 \rfloor} (-1)^k \frac{(2l-2k)!}{2^l k! (l-k)! (l-2k)!} x^{l-2k}, \quad l \in \mathbb{N}$$

$$P_l(1) = 1$$

$$\lambda = l(l+1), \quad y = C_1 P_l(x) + C_2 Q_l(x), \quad Q_l(x) \text{ 有 } \ln(1 \mp x) \text{ 奇性}$$

§10.3 正则奇点邻域的级数解法

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \quad (10.1)$$

$$p(x) = \sum_{k=-\infty}^{\infty} p_k (x-x_0)^k, \quad q(x) = \sum_{k=-\infty}^{\infty} q_k (x-x_0)^k$$

$$y_1(x) = \sum_{k=-\infty}^{\infty} a_k (x-x_0)^{k+s_1}, \quad y_2(x) = \sum_{k=-\infty}^{\infty} b_k (x-x_0)^{k+s_2} + \beta y_1(x) \ln(x-x_0)$$

正则解:

$$y_1(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^{k+s_1} \quad (a_0 \neq 0)$$

$$y_2(x) = \sum_{k=0}^{\infty} b_k (x-x_0)^{k+s_2} + \beta y_1(x) \ln(x-x_0) \quad (b_0 \neq 0 \text{ 或 } \beta \neq 0)$$

Thm (Fuchs) 式(10.1)在 x_0 处有两个正则解的必要条件是:

$(x-x_0)p(x)$ 和 $(x-x_0)^2q(x)$ 在 x_0 解析

正则奇点 (regular singular point)

- ① s_1, s_2 : 指标 $\operatorname{Re} s_1 > \operatorname{Re} s_2$
- ② $s_1 - s_2 \neq 0, 1, 2, \dots$ 第二解必定不含对数 $\beta = 0$
- ③ $s_1 = s_2$ 第二解必含对数 $\beta \neq 0$
- ④ $s_1 - s_2 \in \mathbb{N}^+$ 第二解可能含对数, 也可能不含 $b_0 \neq 0$

$y \sim (x-x_0)^s$ 代入方程

$$\text{令领头阶: } s(s-1)(x-x_0)^{s-2} + p_1 s(x-x_0)^{s-2} + q_2(x-x_0)^{s-2} = 0$$

$$s(s-1) + p_2 s + q_2 = 0 \Rightarrow s_1, s_2$$

先不考虑对数项, 如果得不到线性无关的第二解, 再加入对数项

§10.4 Bessel 方程

§10.4.1 Bessel 方程的级数解

$$y'' + \frac{1}{x} y' + \left(1 - \frac{v^2}{x^2}\right) y = 0$$

v : Bessel 方程的阶, 可以是复数

$$p(x) = \frac{1}{x}, \quad q(x) = 1 - \frac{v^2}{x^2}$$

$x=0$ 是正则奇点

$0, \infty$: 奇点

$$y = x^s \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+s}$$

$$y'' = \sum_{k=0}^{\infty} (k+s)(k+s-1) a_k x^{k+s-2}$$

$$x^2 y'' = \sum_{k=0}^{\infty} (k+s)(k+s-1) a_k x^{k+s}$$

$$x y' = \sum_{k=0}^{\infty} (k+s) a_k x^{k+s}$$

$$x^2 y = \sum_{k=0}^{\infty} a_k x^{k+s+2}$$

$$-v^2 y = -\sum_{k=0}^{\infty} v^2 a_k x^{k+s}$$

$$x^2 y'' + x y' + (x^2 - v^2) y$$

$$= \sum_{k=0}^{\infty} \{ (k+s)^2 - v^2 \} a_k x^{k+s} + \sum_{k=0}^{\infty} a_k x^{k+s+2}$$

$$= (s^2 - v^2) a_0 x^s + \{ (s+1)^2 - v^2 \} a_1 x^{s+1}$$

$$+ \sum_{k=2}^{\infty} \{ (k+s)^2 - v^2 \} a_k x^{k+s} + \sum_{k=0}^{\infty} a_k x^{k+s+2}$$

$$l+s = k+s+2 \Rightarrow k = l-2$$

$$= (s^2 - v^2) a_0 x^s + [(s+1)^2 - v^2] a_1 x^{s+1} + \sum_{k=0}^{\infty} [(k+s+2)^2 - v^2] a_{k+2} + a_k x^{k+s+2}$$

$$a_0 \neq 0, \quad s^2 - v^2 = 0 \Rightarrow s_1 = v, \quad s_2 = -v$$

$s_1 - s_2 = 2v$ 若 v 不等于整数或半整数, 无对数

$$s_1 = v:$$

$$a_1 = 0$$

$$a_{k+2} = - \frac{a_k}{(k+v+2)^2 - v^2} = - \frac{a_k}{(k+2v+2)(k+2)}$$

$$a_{2k+1} = 0, \quad k \in \mathbb{N}$$

$$a_{2k} = (-1)^k \frac{a_0 \Gamma(v+1)}{2^{2k} k! \Gamma(v+k+1)}$$

$$y_1(x) = \sum_{k=0}^{\infty} (-1)^k \frac{a_0 \Gamma(v+1)}{2^{2k} k! \Gamma(v+k+1)} x^{2k+v}$$

$$\text{取 } a_0 = \frac{1}{2^v \Gamma(v+1)}$$

$$J_v(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+v+1)} \left(\frac{x}{2}\right)^{2k+v}$$

$$S_2 = -V:$$

将 ν 换为 $-\nu$, 仍成立

$$J_{-\nu}(x)$$

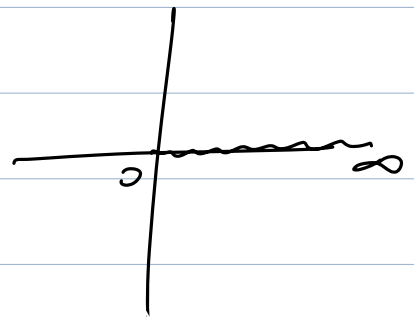
$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x)$$

$x=0$ 支点

$$J_\nu = x^\nu (\dots)$$

$$J_{-\nu} = x^{-\nu} (\dots)$$

收敛半径 ∞



$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt \quad (x > 0)$$

$$\Gamma(1) = 1$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(x+1) = x \Gamma(x)$$

$$\Gamma(n+1) = n!$$

$$\Gamma(0), \Gamma(-1), \dots = \infty$$

$$\Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

$$\psi(z) = \frac{d \ln \Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}$$

§10.4.2 半奇数阶 Bessel 方程

J_ν 和 $J_{-\nu}$ 仍线性独立
为初等函数

§10.4.3 整数阶 Bessel 方程

$$\nu = m \in \mathbb{N}, \quad s_1 - s_2 = 2m$$

$m=0$: 第一解 $y_1(x) = J_0(x)$, 第二解必包含对数

$$m \in \mathbb{N}^+: y_1(x) = J_m(x)$$

$$\nu = m, s_2 = -m \Rightarrow (k+2)(k-2m+2)a_{k+2} = -a_k$$

$$y_2(x) = \beta J_m(x) \ln x + \sum_{k=0}^{\infty} a_k x^{k-m}$$

$$y_2(x) = N_m(x)$$

Karl Neumann 1832-1925

John von Neumann 1903-1957

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2k+m}$$

$J_{-m}(x)$: $k \leq m-1$ 时求和项为零, 求和从 $k=m$ 开始

$$J_{-m}(x) = \sum_{k=m}^{\infty} \frac{(-1)^k}{k! (k-m)!} \left(\frac{x}{2}\right)^{2k-m} = (-1)^m \sum_{k'=0}^{\infty} \frac{(-1)^{k'}}{k'! (k'+m)!} \left(\frac{x}{2}\right)^{2k'+m}$$

$\uparrow$
 $k = k' + m$

$$J_{-m}(x) = (-1)^m J_m(x)$$

$$N_\nu(x) = \frac{J_\nu(x) \cos \nu \pi - J_{-\nu}(x)}{\sin \nu \pi}$$

$\nu \rightarrow N$ 时, $\frac{0}{0}$ 用 L'Hospital 法则

$$y(x) = C_1 J_\nu(x) + C_2 N_\nu(x)$$

$\nu \in \mathbb{Z}$ 类似

$$ay'' + by' + cy = 0$$

$$y = e^{\nu x} \quad \text{代入得}$$

$$a\nu^2 + b\nu + c = 0 \Rightarrow \nu_1, \nu_2$$

$$y = C_1 e^{\nu_1 x} + C_2 e^{\nu_2 x} \quad (\nu_1 \neq \nu_2)$$

$$y = C_1 e^{\nu_1 x} + C_2 \frac{e^{\nu_1 x} - e^{\nu_2 x}}{\nu_1 - \nu_2} \quad (\nu_1 \rightarrow \nu_2)$$

$$\rightarrow C_1 e^{\nu_1 x} + C_2 x e^{\nu_1 x}$$

§ 10.5 Sturm-Liouville 本征值问题

§ 10.5.1 Sturm-Liouville 本征值问题的一般提法

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

$$y''(x) + p(x)y'(x) - \tilde{q}(x)y(x) + \lambda p(x)y(x) = 0$$

$$\frac{d}{dx} \left[k(x) \frac{dy(x)}{dx} \right] - q(x)y(x) + \lambda p(x)y(x) = 0$$

$$ky'' + k'y' + \dots = 0$$

$$\Rightarrow \frac{k'}{k} = \bar{p}(x) \Rightarrow k(x) = e^{\int_{x_0}^x p(\xi) d\xi}$$

λ : eigenvalue, y : eigenfunction

1. 第一、第二、第三类边界条件

$$\begin{cases} y'' + \lambda y = 0 \\ (\alpha y' - \beta y)|_{x=a} = 0, \quad (\gamma y' + \delta y)|_{x=b} = 0 \end{cases}$$

$\alpha, \beta, \gamma, \delta \geq 0$, α 和 β 不同时为 0, γ 和 δ 不同时为 0

$$k(x) = 1, \quad q(x) = 0, \quad p(x) = 1$$

2. 自然边界条件

$$\begin{cases} \frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] + \lambda y = 0 \\ y(\pm 1) < \infty \end{cases} \quad \text{Legendre eq.}$$

$$\begin{cases} \frac{d}{dx} \left(x \frac{dy}{dx} \right) - \frac{m^2}{x} + \lambda xy = 0 \\ \dots \end{cases} \quad \text{Bessel eq.}$$

$$y'' - 2xy' + \lambda y = 0 \quad \text{Hermite eq.}$$

一般来说, 端点 a 处出现自然边界条件的必要条件是 $k(a) = 0$
 b $k(b) = 0$

$$y''(x) + \frac{k'(x)}{k(x)} y'(x) + \frac{\lambda p(x) - q(x)}{k(x)} y(x) = 0$$

3. 周期性边界条件

如果 $k(a) = k(b)$, $q(a) = q(b)$, $p(a) = p(b)$, 可加周期性 BC: $y(a) = y(b)$
 $y'(a) = y'(b)$

$$\begin{cases} y'' + \lambda y = 0 & (0 < x < 2\pi) \\ y(0) = y(2\pi), \quad y'(0) = y'(2\pi) \end{cases}$$

§ 10.5.2 Sturm-Liouville 本征值问题的一般结论

$$\frac{d}{dx} \left[k(x) \frac{dy}{dx} \right] - q(x) y(x) + \lambda p(x) y(x) = 0$$

$$k(x) \geq 0, \quad q(x) \geq 0, \quad p(x) \geq 0$$

1. 所有本征值都是非负的, 即 $\lambda \geq 0$

$$\text{证: } y^*(x) \frac{d}{dx} \left[k(x) \frac{dy}{dx} \right] - q(x) y^*(x) y(x) + \lambda p(x) y^*(x) y(x) = 0$$

$$\begin{aligned} \lambda \int_a^b p(x) |y(x)|^2 dx &= \int_a^b q(x) |y(x)|^2 dx - \int_a^b y^*(x) \frac{d}{dx} \left[k(x) \frac{dy}{dx} \right] dx \\ &= \int_a^b q(x) |y(x)|^2 dx + \int_a^b k(x) |y'(x)|^2 dx - k(x) y^*(x) y(x) \Big|_a^b \\ &\geq k(a) y^*(a) y'(a) - k(b) y^*(b) y'(b) \end{aligned}$$

第一类 BC: $y(a) = 0$

$$= y'(a) = 0$$

$$= \alpha y'(a) - \beta y(a) = 0 \Rightarrow k(a) y^*(a) y'(a) = \frac{\beta}{\alpha} k(a) |y(a)|^2 \geq 0$$

自然 BC: $k(a) = 0$

周期性 BC: $y(a) = y(b)$

2. 存在无穷多分立的本征值: $\lambda_1 < \lambda_2 < \dots < \lambda_n < \lambda_{n+1} < \dots$, 且 $\lim_{n \rightarrow \infty} \lambda_n = +\infty$. 除了周期性边界条件的情况, 本征值都是非简并的, 且 $y_{n+1}(x)$ 比 $y_n(x)$ 多一个零点.
 $\downarrow$ 简并度: 2

3. 对应于不同本征值的本征函数在区间 $[a, b]$ 上带权正交:

$$\int_a^b y_m^*(x) y_n(x) \rho(x) dx = 0, \quad (\lambda_m \neq \lambda_n). \quad (115)$$

注 ① 本征函数族的正交性对于后面计算广义 Fourier 级数的系数非常重要. 有时候, 通过直接计算来验证本征函数族的正交性有一定的困难, 所以上述结论给我们带来了很大的方便. ② 与三角函数族的正交性相比, 这里有两点推广: 一是多了权函数 $\rho(x)$, 如果 $\rho(x) = 1$, 就是普通正交; 二是考虑了本征函数是复值函数的情况 (注意自变量仍是实数, 只是函数值可取复数, 故并非复变函数), 比如本征值问题 (114) 的本征函数族 $\{e^{imx}\}$, 其中 $0 \leq x \leq 2\pi$, 而 $m = 0, \pm 1, \pm 2, \dots$. ③ 对应于同一本征值的两个本征函数 (如果有简并) 不一定相互正交, 但我们总可以取其适当的线性组合 (线性组合后的函数仍是对应于同一本征值的本征函数) 使得它们相互正交, 通过这样的做法 (称为 Schmidt 正交化), 可以使所有的本征函数相互正交.

$$y_m^*(x) \frac{d}{dx} \left\{ k(x) \frac{dy_n(x)}{dx} \right\} - q(x) y_m^*(x) y_n(x) + \lambda_n \rho(x) y_m^*(x) y_n(x) = 0 \quad (1)$$

$$y_n(x) \frac{d}{dx} \left\{ k(x) \frac{dy_m^*(x)}{dx} \right\} - q(x) y_n(x) y_m^*(x) + \lambda_m \rho(x) y_n(x) y_m^*(x) = 0 \quad (2)$$

① - ②

$$\begin{aligned} (\lambda_m - \lambda_n) \int_a^b \rho(x) y_m^*(x) y_n(x) dx &= \int_a^b \{ y_m^* [k y_n']' - y_n [k y_m'^*]' \} dx \\ &= k(x) [y_m^* y_n' - y_m'^* y_n] \Big|_a^b = 0 \end{aligned}$$

$$\gamma y_n'(b) + \delta y_n(b) = 0$$

$$\gamma y_m'^*(b) + \delta y_m^*(b) = 0$$

$$\begin{pmatrix} y_n'(b) & y_n(b) \\ y_m'^*(b) & y_m^*(b) \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = 0$$

$$\det M = 0 \Rightarrow y_n^*(b) y_n'(b) - y_m'^*(b) y_n(b) = 0$$

4. 本征函数族 $\{y_n(x)\}_{n=1}^{\infty}$ 在区间 $[a, b]$ 上是完备的。这就是说，区间 $[a, b]$ 上的任意函数 $f(x)$ ，只要解析性质良好且与本征函数族满足相同的边界条件，就一定可以用 $\{y_n(x)\}_{n=1}^{\infty}$ 展开为广义 Fourier 级数：

$$f(x) = \sum_{n=1}^{\infty} f_n y_n(x), \quad \Leftrightarrow \quad \vec{A} = \sum_i A_i \vec{e}_i$$

其中展开系数为

$$f_n = \frac{\int_a^b y_n^*(x) f(x) \rho(x) dx}{\int_a^b y_n^*(x) y_n(x) \rho(x) dx}, \quad n \in \mathbb{N}^+. \quad \Leftrightarrow \quad A_i = \frac{1}{|\vec{e}_i|^2} \vec{A} \cdot \vec{e}_i$$

注 我们已经知道，本征函数族的完备性是分离变量法的理论基础。所以，这一结论的重要性是显而易见的。

"世界是稳定的"

能量有下界

能量有可加性

Mathematical

Bessel 方程有 $\ln$ 的情况

Chap. 11 球函数

§11.1 轴对称球函数

$$\nabla^2 u = 0$$

(回顾)

$$u = R(r) \underbrace{H(\theta) \Phi(\varphi)}_{Y(\theta, \varphi)}$$

$$\Phi(\phi) = \{e^{im\phi}, e^{-im\phi}\}, \quad m \in \mathbb{N}$$

$$\begin{cases} \frac{d}{dx}[(1-x^2) \frac{dP}{dx}] + (\lambda - \frac{m^2}{1-x^2}) P = 0 \\ P(\pm 1) = 0 \quad (m \neq 0) \quad \text{或} \quad |P(\pm 1)| < \infty \quad (m = 0) \end{cases}$$

轴对称: 解与 ϕ 无关, $m = 0$

$$\begin{cases} \frac{d}{dx}[(1-x^2) \frac{dP}{dx}] + \lambda P = 0 \\ |P(\pm 1)| < \infty \end{cases}$$

$$\lambda = l(l+1), \quad P(x) = P_l(x), \quad l \in \mathbb{N}$$

$$r^2 R''(r) + 2r R'(r) - \lambda R(r) = 0$$

$$\Rightarrow R(r) = \{r^l, \frac{1}{r^{l+1}}\}$$

轴对称问题的一般解:

球对称, 柱对称, 轴对称

$$u(r, \theta) = \sum_{l=0}^{\infty} (A_l r^l + \frac{B_l}{r^{l+1}}) P_l(\cos \theta)$$

§11.1.2 Legendre 多项式的基本性质

Mathematica

$$P_\ell(x) = \sum_{k=0}^{\lfloor \ell/2 \rfloor} (-1)^k \frac{(2\ell-2k)!}{2^\ell k! (\ell-k)! (\ell-2k)!} x^{\ell-2k}, \quad \ell \in \mathbb{N}$$

1. 奇偶性: $P_\ell(-x) = (-1)^\ell P_\ell(x)$

2. 原点值:

$$P_{2n+1}(0) = 0, \quad P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n} (n!)^2}, \quad n \in \mathbb{N}$$

↓
 $P_{2n}(x)$ 的常数项, $k=2n$ ($\ell=2n$)

3. 端点值:

$$P_\ell(1) = 1, \quad P_\ell(-1) = (-1)^\ell$$

4. $P_0(x) = 1$

$$P_1(x) = x$$

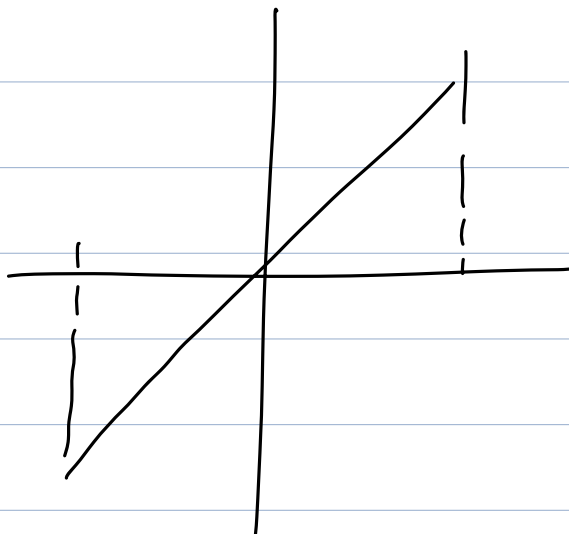
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

5. 零点 $\ell \uparrow$

图:



§11.1.3 Legendre 多项式的微分表示

differential representation

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$$

证明 由二项式定理,

$$(x^2 - 1)^l = \sum_{k=0}^l C_l^k x^{2l-2k} (-1)^k = \sum_{k=0}^l (-1)^k C_l^k x^{2l-2k},$$

将上式对 x 求导 l 次, 容易看出, 只有满足 $2l - 2k \geq l$ 的项求导后才不为零, 换句话说, 求导后只需对 $k \leq l/2$ 求和, 利用求导公式 $(d^l/dx^l)x^n = [n!/(n-l)!]x^{n-l}$, 可得

$$\begin{aligned} \frac{d^l}{dx^l} (x^2 - 1)^l &= \sum_{k=0}^{[l/2]} (-1)^k C_l^k \frac{d^l}{dx^l} x^{2l-2k} = \sum_{k=0}^{[l/2]} (-1)^k C_l^k \frac{(2l-2k)!}{(l-2k)!} x^{l-2k} \\ &= \sum_{k=0}^{[l/2]} (-1)^k \frac{l!(2l-2k)!}{k!(l-k)!(l-2k)!} x^{l-2k}, \end{aligned}$$

所以

$$\frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l = \sum_{k=0}^{[l/2]} (-1)^k \frac{(2l-2k)!}{2^l k!(l-k)!(l-2k)!} x^{l-2k},$$

$$P_l(1) = 1$$

§11.1.4 * 积分表示

§11.1.5 Legendre 多项式的正交关系

§11.1.6 - - - - - 模

$$\int_{-1}^1 P_l(x) P_l(x) dx = \|P_l\|^2 \delta_{ij}$$

正交关系 只需证 $\int_{-1}^1 x^k P_l(x) dx = 0 \quad (k < l)$

$$\text{设 } l' < l, \quad P_l(x) P_{l'}(x) = \sum_{k=0}^{l'} \# x^k P_l(x) \quad (k < l)$$

$$\int_{-1}^1 x^k P_l(x) dx = \frac{1}{2^l l!} \int_{-1}^1 x^k \frac{d^l}{dx^l} (x^2 - 1)^l dx$$

$$= \frac{1}{2^l l!} x^k \cdot \frac{d^{l-1}}{dx^{l-1}} (x^2 - 1)^l \Big|_{-1}^1 - \frac{1}{2^l l!} \int_{-1}^1 \frac{d}{dx} x^k \cdot \frac{d^{l-1}}{dx^{l-1}} (x^2 - 1)^l dx$$

$$= \dots$$

$$= \frac{(-1)^l}{2^l l!} \int_{-1}^1 \left(\frac{d^l}{dx^l} x^k \right) \cdot (x^2 - 1)^l dx = 0 \quad (k < l)$$

利用 Legendre 多项式的微分表示, 有

$$\begin{aligned} \|P_l\|^2 &= \int_{-1}^1 [P_l(x)]^2 dx = \frac{1}{2^l l!} \int_{-1}^1 P_l(x) \frac{d^l}{dx^l} (x^2 - 1)^l dx = \frac{1}{2^l l!} \int_{-1}^1 P_l(x) d \left[\frac{d^{l-1}}{dx^{l-1}} (x^2 - 1)^l \right] \\ &= \frac{1}{2^l l!} P_l(x) \frac{d^{l-1}}{dx^{l-1}} (x^2 - 1)^l \Big|_{-1}^1 - \frac{1}{2^l l!} \int_{-1}^1 P'_l(x) \frac{d^{l-1}}{dx^{l-1}} (x^2 - 1)^l dx \\ &= -\frac{1}{2^l l!} \int_{-1}^1 P'_l(x) \frac{d^{l-1}}{dx^{l-1}} (x^2 - 1)^l dx, \end{aligned}$$

其中作了分部积分, 而积出部分为零, 这是由于

$$\begin{aligned} \frac{d^{l-1}}{dx^{l-1}} (x^2 - 1)^l &= \frac{d^{l-1}}{dx^{l-1}} [(x-1)^l (x+1)^l] = \sum_{k=0}^{l-1} C_{l-1}^k \left[\frac{d^k}{dx^k} (x-1)^l \right] \left[\frac{d^{l-1-k}}{dx^{l-1-k}} (x+1)^l \right] \\ &= \sum_{k=0}^{l-1} C_{l-1}^k \frac{l!}{(l-k)!} \frac{l!}{(k+1)!} (x-1)^{l-k} (x+1)^{k+1}, \end{aligned}$$

注意到 k 的取值范围, 易见上式在 $x = \pm 1$ 处均为零. 继续作分部积分, 共作 l 次, 可以得到

$$\|P_l\|^2 = \frac{(-1)^l}{2^l l!} \int_{-1}^1 \frac{d^l P_l(x)}{dx^l} (x^2 - 1)^l dx. \quad (25)$$

由于 $P_l(x)$ 是 l 次多项式, 求导 l 次后为常数, 事实上,

$$\frac{d^l P_l(x)}{dx^l} = \frac{1}{2^l l!} \frac{d^{2l}}{dx^{2l}} (x^2 - 1)^l = \frac{1}{2^l l!} \frac{d^{2l}}{dx^{2l}} x^{2l} = \frac{(2l)!}{2^l l!},$$

其中幂次小于 $2l$ 的各项求导后为零, 代入式 (25) 得到

$$\|P_l\|^2 = \frac{(-1)^l (2l)!}{2^{2l} (l!)^2} \int_{-1}^1 (x^2 - 1)^l dx = \frac{2(2l)!}{2^{2l} (l!)^2} \int_0^1 (1 - x^2)^l dx. \quad (26)$$

$$\int_{-1}^1 P_\ell(x) P_{\ell'}(x) dx = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

§ 11.1.7 广义 Fourier 级数

$$f(x) = \sum_{\ell=0}^{\infty} f_\ell P_\ell(x), \quad -1 \leq x \leq 1$$

$$f_\ell = \frac{2\ell+1}{2} \int_{-1}^1 f(x) P_\ell(x) dx, \quad \ell = 0, 1, 2, \dots$$

例:
$$x^n = \sum_{k=0}^{[n/2]} \frac{(2n-4k+1)2^{n-2k}n!(n-k)!}{k!(2n-2k+1)!} P_{n-2k}(x).$$

$$|x| = \frac{1}{2} P_0(x) + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(4n+1)(2n-1)!!}{(2n-1)(2n+2)!!} P_{2n}(x), \quad -1 \leq x \leq 1$$

§ 11.1.8 Legendre 多项式的母函数

$$\nabla^2 U(\vec{r}) = -4\pi \delta(x) \delta(y) \delta(z-1)$$

$$\nabla^2 U(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}$$

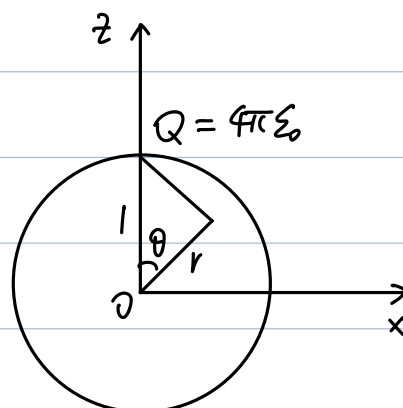
$$\rho(\vec{r}) = 4\pi \epsilon_0 \delta(x) \delta(y) \delta(z-1)$$

$$U(\vec{r}) = \frac{Q}{4\pi \epsilon_0} \frac{1}{|\vec{r} - \vec{r}_0|}$$

$$= \frac{1}{\sqrt{1-2r \cos \theta + r^2}}$$

$$U(\vec{r}) = U(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

$(r < 1)$



$$\frac{1}{\sqrt{1-2rx \cos \theta + r^2}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

$(r < 1)$

$$x=1: \quad \frac{1}{1-r} = \sum_{l=0}^{\infty} A_l r^l \quad \Rightarrow \quad A_l = 1 \quad (\text{all } l)$$

$$\frac{1}{\sqrt{1-2rx+r^2}} = \sum_{l=0}^{\infty} r^l P_l(x) \quad (r < 1)$$

$$\frac{1}{\sqrt{1-2rx+r^2}} = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} P_l(x) \quad (r > 1)$$

§ 11.1.9 Legendre 多项式的递推关系

$$(2l+1)x P_l(x) = (l+1) P_{l+1}(x) + l P_{l-1}(x)$$

$$(2l+1) P_l(x) = P'_{l+1}(x) - P'_{l-1}(x)$$

很多

§11.1.10 应用

例1: 已知半径为 a 的球面上的电势分布为 $f(\theta)$, 球内外无电荷, 求空间各处的电势.

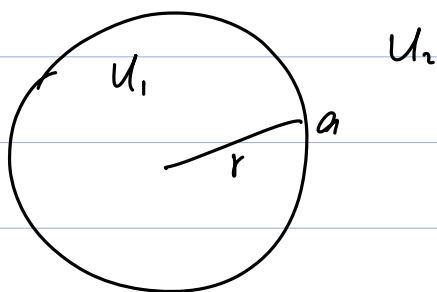
↓
零点取在 ∞

↓ Laplace 方程

$$\begin{cases} \nabla^2 u = 0 & (r < a, r > a) \\ u|_{r=a} = f(\theta), & u|_{r=\infty} = 0 \end{cases}$$

定解条件与 φ 无关: 轴对称问题 - 一般解

$$u = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$



$$u_1 = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

代入 $r=a$ 边界条件: $\sum_{l=0}^{\infty} A_l a^l P_l(\cos \theta) = f(\theta)$

$$\sum_{l=0}^{\infty} A_l a^l P_l(x) = g(x) \quad \swarrow x = \cos \theta$$

$$A_l a^l = \frac{2l+1}{2} \int_{-1}^1 g(x) P_l(x) dx = \frac{2l+1}{2} \int_0^\pi f(\theta) P_l(\cos \theta) \sin \theta d\theta$$

$\Rightarrow A_l$ 得球内电势 u_1

$$u_2 = \sum_{l=0}^{\infty} \frac{D_l}{r^{l+1}} P_l(\cos \theta)$$

代入 $r=a$ 边界条件: $\sum_{l=0}^{\infty} \frac{D_l}{a^{l+1}} P_l(\cos \theta) = f(\theta)$

$$\frac{D_l}{a^{l+1}} = A_l a^l$$

特例: $f(\theta) = \cos^2 \theta = x^2$

$$P_0 = 1, \quad P_1 = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$x^2 = \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x)$$

$$A_0 = \frac{1}{3}, \quad A_2 a^2 = \frac{2}{3}$$

$$A_i = 0 \quad (i \neq 0, 2)$$

$$\frac{D_0}{a} = \frac{1}{3}, \quad \frac{D_2}{a^3} = \frac{2}{3}$$

$$D_i = 0 \quad (i \neq 0, 2)$$

球内: $u_1(r, \theta) = \frac{1}{3} + \frac{2}{3a^2} \cdot r^2 P_2(\cos \theta)$

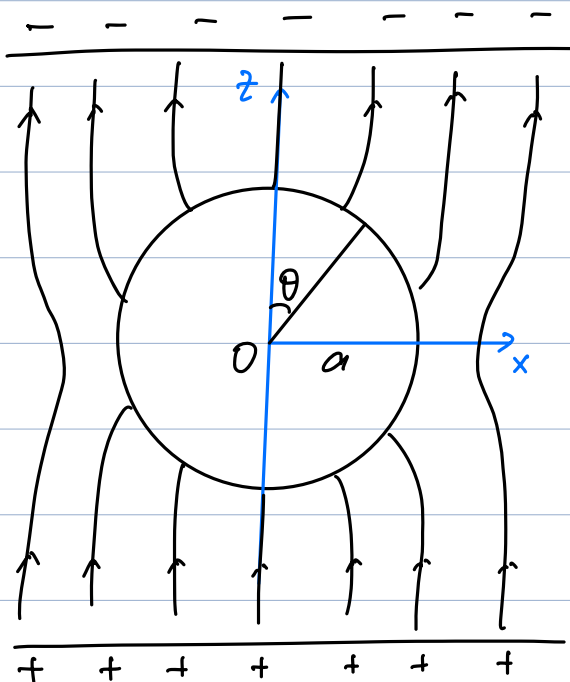
球外: $u_2(r, \theta) = \frac{a}{3} \cdot \frac{1}{r} + \frac{2a^3}{3} \cdot \frac{1}{r^3} P_2(\cos \theta)$

例2: 在均匀电场 $\vec{E}_0$ 中放入半径为 a , 带电为 Q 的导体球, 求放入导体球后的电场分布.

1. 物理分析

2. 建立坐标系

3. 建立定解问题



$$U(\vec{r}) = U(r, \theta)$$

$$\left\{ \begin{array}{l} \nabla^2 U = 0 \quad (r > a) \\ U|_{r=a} = U_0 \\ \int_{r=a^+} \left(-\frac{\partial U}{\partial r} \right) dS = \frac{Q}{\epsilon_0} \\ U|_{r \rightarrow \infty} = -E_0 r \cos \theta \end{array} \right.$$

$$U = -E_0 z$$

4. 求解定解问题

$$U = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

$$\vec{A} = \vec{B} \Rightarrow A_l = B_l$$

$$\begin{aligned} \text{fit } U|_{r \rightarrow \infty} &\Rightarrow \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) = -E_0 r \cos \theta = -E_0 r P_1(\cos \theta) \\ &\Rightarrow A_1 = -E_0, \quad A_l = 0 \quad (l \neq 1) \end{aligned}$$

$$U(r, \theta) = -E_0 r \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

$$\text{fit } U|_{r=a} \Rightarrow -E_0 a P_1(\cos \theta) + \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos \theta) = U_0 = U_0 P_0(\cos \theta)$$

$$\Rightarrow B_0 = U_0 a, \quad B_1 = E_0 a^3, \quad B_l = 0 \quad (l \neq 0, 1)$$

$$u(r, \theta) = -E_0 r \cos \theta + \frac{E_0 a^3}{r^2} \cos \theta + \frac{U_0 a}{r}$$

球面上积分 $dS = a^2 \sin \theta d\theta d\varphi$

$$\int \cos \theta \cdot a^2 \sin \theta d\theta d\varphi = 0$$

$$\int_{r=a^+} \left(-\frac{\partial u}{\partial r}\right) dS = \int_{r=a^+} \left(-\frac{U_0 a}{r^2}\right) a^2 \sin \theta d\theta d\varphi = 4\pi U_0 a = \frac{Q}{\epsilon_0}$$

$$u(r, \theta) = -E_0 r \cos \theta + \frac{E_0 a^3}{r^2} \cos \theta + \frac{Q}{4\pi \epsilon_0 r}$$

5. 结果分析

原匀电场

电偶极子

导体球带电量 Q

$Q = 0$:

$$E_r = -\left.\frac{\partial u}{\partial r}\right|_{r=a} = 3E_0 \cos \theta, \quad E_\theta = -\left.\frac{1}{r} \frac{\partial u}{\partial \theta}\right|_{r=a} = 0.$$

$$\sigma(\theta) = \epsilon_0 E_r|_{r=a} = 3\epsilon_0 E_0 \cos \theta.$$

本题的结果在 $r \rightarrow \infty$ 时有奇性。这是由于无穷远处有源

感应电荷的分布对于球外来说等价于一个理想电偶极子。

§ 11.2 一般球函数

§ 11.2.1 非轴对称问题, 连带 Legendre 函数

$$\frac{d}{dx} \left[(1-x^2) \frac{dP}{dx} \right] + \left(\lambda - \frac{m^2}{1-x^2} \right) P = 0 \quad (*)$$

$$P(\pm 1) = 0 \quad (m \neq 0) \quad \text{或} \quad |P(\pm 1)| < \infty \quad (m = 0)$$

$$P(x) = (1-x^2)^{\frac{m}{2}} \gamma(x)$$

代入 (*) 得

$$(1-x^2) \gamma'' - 2(m+1)x\gamma' + [\lambda - m(m+1)]\gamma = 0 \quad \left. \begin{array}{l} \text{求 } m \text{ 次} \end{array} \right\}$$

$$(1-x^2) f'' - 2xf' + \lambda f = 0$$

$$\gamma = f^{(m)}(x)$$

$$\lambda = l(l+1), \quad l = m, m+1, \dots$$

$$P_l^m(x) = (1-x^2)^{\frac{m}{2}} P_l^{(m)}(x) \quad Q_l^m(x)$$

连带 Legendre 函数 (Associated Legendre function)

m 事先给定, $l = m, m+1, \dots$

$x = \pm 1$ 是 (*) 的正则奇点, 两点的指标都是 $\pm \frac{m}{2}$

$$\gamma(x) \sim (1 \pm x)^0, (1 \pm x)^{-m}$$

§ 11.2.2 连带 Legendre 函数的微分表示

$$P_l^m(x) = \frac{1}{2^l l!} (1-x)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l$$

§11.2.3 函数 $P_l^m(x)$

$m \rightarrow -m$ 方程不变, $P_l^{-m}(x) \propto P_l^m(x)$

$$P_l^m(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$$

§11.2.4 连带 Legendre 函数的正交关系

§11.2.5 - - - - - 模

$$\int_{-1}^1 P_l^m(x) P_l^m(x) dx = \frac{(l+m)!}{(l-m)!} \frac{2}{2l+1} \delta_{ll'}$$

§11.2.6 广义 Fourier 级数

作为 Sturm-Liouville 本征值问题的特例, 连带 Legendre 函数在区间 $[-1, 1]$ 上是完备的, 所以, 区间 $[-1, 1]$ 上的任一函数 $f(x)$, 如果具有良好的解析性质, 必可用 $\{P_l^m(x)\}_{l=m}^{\infty}$ 展开为广义 Fourier 级数

$$f(x) = \sum_{l=m}^{\infty} f_l P_l^m(x), \quad -1 \leq x \leq 1, \quad \Leftrightarrow \vec{A} = \sum_i A_i \vec{e}_i$$

$$f_l = \frac{1}{\|P_l^m\|^2} \int_{-1}^1 f(x) P_l^m(x) dx, \quad l = m, m+1, m+2, \dots \Leftrightarrow A_i = \frac{1}{\|\vec{e}_i\|^2} \vec{A} \cdot \vec{e}_i$$

§11.3.7 连带 Legendre 函数的递推关系

§11.3.8 球谐函数

Spherical harmonics

Laplace 方程或 Helmholtz 方程在球坐标系中作分离变量 $u(\mathbf{r}) = R(r)Y(\theta, \phi)$ 后, 角向部分 $Y(\theta, \phi)$ 满足球函数方程

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} + \lambda Y = 0. \quad (84a)$$

由于球坐标系的特点, 函数 $Y(\theta, \phi)$ 应该满足以下两个自然边界条件

$$Y(0, \phi), \quad Y(\pi, \phi) \quad \text{没有奇性}, \quad (84b)$$

$$Y(\theta, \phi + 2\pi) = Y(\theta, \phi). \quad (84c)$$

这里没有奇性指的是具有确切的定义 (well defined), 并且是有限的. 上述方程和边界条件构成了一个本征值问题, 不过是 偏微分方程的本征值问题.

$Y(\theta, \phi) = H(\theta) \Phi(\phi)$ 之前已解出

本征值 $\lambda_l = l(l+1)$

本征函数 $P_l^m(\cos \theta) \{e^{im\phi}, e^{-im\phi}\} \quad m=0, 1, 2, \dots, l$

$P_l^m(\cos \theta) e^{im\phi} \quad m=0, \pm 1, \pm 2, \dots, \pm l$

一个本征值对应 $2l+1$ 个本征函数 简并度 $2l+1$

对比 ODE

物理上更常用的是归一化的球谐函数

$$Y_{lm}(\theta, \phi) = (-)^m \frac{P_l^m(\cos \theta)}{\|P_l^m\|} \frac{e^{im\phi}}{\sqrt{2\pi}} = (-)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi},$$
$$l \in \mathbb{N}; \quad m = 0, \pm 1, \pm 2, \dots, \pm l. \quad (86)$$

以上球谐函数在单位球面上满足正交归一关系 (简称正一关系, orthonormal relation)

$$\int_0^{2\pi} \int_0^\pi Y_{l'm'}^*(\theta, \phi) Y_{lm}(\theta, \phi) \sin \theta d\theta d\phi = \delta_{ll'} \delta_{mm'}. \quad (87)$$

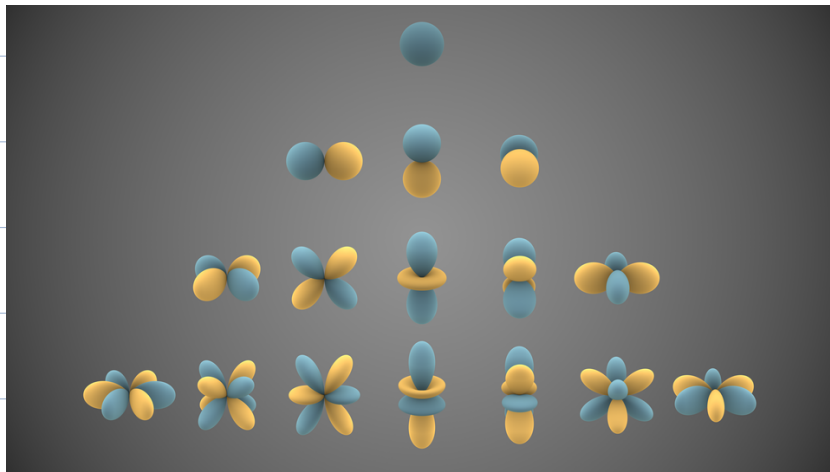
球谐函数在球面上是完备的，这是因为 $\{e^{im\phi}\}_{m=-\infty}^{\infty}$ 在区间 $\phi \in [0, 2\pi]$ 上是完备的，而 $\{P_l^m(\cos \theta)\}_{l=|m|}^{\infty}$ 在区间 $\theta \in [0, \pi]$ 上是完备的。所以，球面上的任意函数 $f(\theta, \phi)$ ，如果具有良好的解析性质，一定可以展开为

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_{lm} Y_{lm}(\theta, \phi), \quad \Leftrightarrow \vec{A} = \sum_i A_i \vec{e}_i$$

$$f_{lm} = \int_0^{2\pi} \int_0^{\pi} Y_{lm}^*(\theta, \phi) f(\theta, \phi) \sin \theta d\theta d\phi. \quad \Leftrightarrow A_i = \vec{A} \cdot \vec{e}_i$$

Laplace 方程在球坐标系下的一般解：

$$u(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right) Y_{lm}(\theta, \varphi)$$



$$l=0 \quad s$$

$$l=1 \quad p$$

$$l=2 \quad d$$

$$l=3 \quad f$$

方程有球对称性 $SO(3)$

$$u(r, \theta, \varphi) \sim R(r) Y_{lm}(\theta, \varphi)$$

对称性 $\Leftrightarrow$ 能级简并

Chap. 13 Green 函数法

§13.1 Green 函数的一般概念

线性偏微分方程

$$Lu = f$$

波动方程: $L = \frac{\partial^2}{\partial t^2} - a^2 \nabla^2$

输运方程: $L = \partial_t - a^2 \nabla^2$

Poisson 方程: $L = \nabla^2$

Helmholtz 方程: $L = -(\nabla^2 + k^2)$

$$Lu = f \Rightarrow u = L^{-1}f$$

例: $\frac{d}{dx} u = f \Rightarrow u = \int f(x) dx$

$$L = \frac{d}{dx}$$

$$L^{-1} = \int (\cdot) dx$$

$$Lu = f$$

$$L^{-1} = \int G(\cdot) dx \quad u = \int G f dx$$

$$u = \int G(\vec{r}, \vec{r}_0) f(\vec{r}_0) d\vec{r}_0$$

— 劳永逸

$$L G(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0)$$

$$Lu = L \int G(\vec{r}, \vec{r}_0) f(\vec{r}_0) d\vec{r}_0 = \int (L G) \cdot f(\vec{r}_0) d\vec{r}_0$$

$$= \int \delta(\vec{r} - \vec{r}_0) f(\vec{r}_0) d\vec{r}_0 = f(\vec{r})$$

边界条件. 适当

点源影响函数, 无界空间的 Green 函数又称为基本解

如何解 $y'' + p(x)y' + q(x)y = \delta(x)$,

$$y''(x) = \delta(x)$$

$$y = \frac{1}{2}|x| + Ax + B$$

例: Poisson 方程 $\nabla^2 u(\vec{r}) = -f(\vec{r})$

$$\nabla^2 G(\vec{r}, \vec{r}_0) = -\delta(\vec{r} - \vec{r}_0)$$

$$r \rightarrow \infty \text{ 时 } G(\vec{r}, \vec{r}_0) \rightarrow 0$$

$$G(\vec{r}, \vec{r}_0) = \frac{1}{4\pi|\vec{r} - \vec{r}_0|}$$

库仑定律

↑ ↑
场点 源点

$$u(\vec{r}) = \int \frac{f(\vec{r}_0)}{4\pi|\vec{r} - \vec{r}_0|} d\vec{r}_0$$

合时问题选读

§13.2 稳定场方程的 Green 函数法

§13.2.1 Poisson 方程的基本解

$$\nabla^2 u(\vec{r}) = -f(\vec{r})$$

$$\nabla^2 G(\vec{r}, \vec{r}_0) = -\delta(\vec{r} - \vec{r}_0)$$

$$3D, G(\vec{r}, \vec{r}_0) \rightarrow 0 \quad (r \rightarrow \infty)$$

$$G(\vec{r}, \vec{r}_0) = \frac{1}{4\pi|\vec{r} - \vec{r}_0|}$$

Pf 令 $\vec{R} = \vec{r} - \vec{r}_0$, $R = |\vec{R}|$, $G(\vec{r}, \vec{r}_0) = \frac{1}{4\pi R}$

$$\left(\begin{array}{l} \delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \\ \int_{-\infty}^{\infty} \delta(x) dx = 1 \end{array} \right)$$

$$\nabla^2 G(\vec{r}, \vec{r}_0) = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial G}{\partial R} \right) + 0 + 0$$

$$= \frac{1}{R^2} \frac{d}{dR} \left(R \cdot \frac{-1}{4\pi R^2} \right) = 0 \quad \vec{R} \neq 0$$

以 $\vec{r}_0$ 为球心、 ε 为半径的球 K_ε , 其表面记作 S_ε

$$\int_{K_\varepsilon} \nabla^2 G(\vec{r}, \vec{r}_0) d\vec{r} = \int_{K_\varepsilon} \nabla \cdot \nabla \left(\frac{1}{4\pi R} \right) d\vec{r} = \int_{S_\varepsilon} \nabla \left(\frac{1}{4\pi R} \right) \cdot d\vec{\sigma} = - \int_{S_\varepsilon} \frac{1}{4\pi \varepsilon^2} d\sigma = -1$$

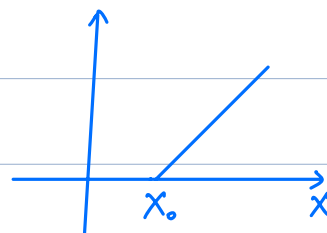
Gauss 定理

$$\iiint_V \nabla \cdot \vec{E} dV = \oint_S \vec{E} \cdot d\vec{S}$$

2D $\nabla^2 G(\vec{r}, \vec{r}_0) = -\delta(\vec{r} - \vec{r}_0)$

$$G(\vec{r}, \vec{r}_0) = -\frac{1}{2\pi} \ln |\vec{r} - \vec{r}_0| = \frac{1}{2\pi} \ln \frac{1}{|\vec{r} - \vec{r}_0|}$$

	势	力
4D	$\frac{1}{r^2}$	$\frac{1}{r^3}$
3D	$\frac{1}{r}$	$\frac{1}{r^2}$
2D	$\ln r$	$\frac{1}{r}$

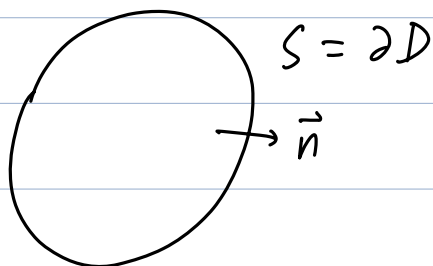


1D $G''(x, x_0) = \delta(x - x_0)$

$$G(x, x_0) = \begin{cases} 0, & x < x_0 \\ x - x_0, & x > x_0 \end{cases}$$

格林公式

§13.2.2 第二格林公式



$$\int_D (u \nabla^2 v - v \nabla^2 u) d\vec{r} = \int_S (u \nabla v - v \nabla u) \cdot d\vec{\sigma} = \int_S \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) d\sigma$$

$$\nabla \cdot (u \nabla v) = \nabla u \cdot \nabla v + u \nabla^2 v \quad ①$$

$$\nabla \cdot (v \nabla u) = \nabla u \cdot \nabla v + v \nabla^2 u \quad ②$$

$$① - ② \Rightarrow u \nabla^2 v - v \nabla^2 u = \nabla \cdot (u \nabla v - v \nabla u)$$

Gauss 定理:

$$\nabla v \cdot d\vec{\sigma} = \frac{\partial v}{\partial n} d\sigma$$

§11.2.3 Poisson 方程第一边值问题

$$\begin{cases} \nabla^2 u(\vec{r}) = -f(\vec{r}), & \vec{r} \in D \\ u(\vec{r})|_S = \varphi(\vec{r}) \end{cases} \quad ①$$

$$\nabla^2 G(\vec{r}, \vec{r}_0) = -\delta(\vec{r} - \vec{r}_0) \quad ②$$

$$G(\vec{r}, \vec{r}_0) \times ① - u(\vec{r}) \times ② \Rightarrow$$

$$G \nabla^2 u - u \nabla^2 G = -Gf + u \delta(\vec{r} - \vec{r}_0)$$

($\vec{r}, \vec{r}_0$)

积分:

$$\int_S \left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n} \right) d\sigma = - \int_D G(\vec{r}, \vec{r}_0) f(\vec{r}) d\vec{r} + u(\vec{r}_0)$$

$$\Rightarrow u(\vec{r}_0) = \int_D G(\vec{r}, \vec{r}_0) f(\vec{r}) d\vec{r} + \int_S \left(\underbrace{G(\vec{r}, \vec{r}_0)}_{\text{未知}} \frac{\partial u(\vec{r})}{\partial n} - \underbrace{u(\vec{r})}_{\text{已知 (第一类边界条件)}} \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n} \right) d\sigma$$

$$G(\vec{r}, \vec{r}_0)|_{r \in S} = 0$$

Green fcn 满足下列定解问题

"辅助线"

$$\begin{cases} \nabla^2 G(\vec{r}, \vec{r}_0) = -\delta(\vec{r}, \vec{r}_0), & r, r_0 \in D \\ G(\vec{r}, \vec{r}_0)|_{r \in S} = 0 \end{cases}$$

$$u(r_0) = \int_D G(\vec{r}, \vec{r}_0) f(\vec{r}) d\vec{r} - \int_S \varphi(\vec{r}) \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n} d\vec{r}$$

$$u(\vec{r}) = \int_D G(\vec{r}, \vec{r}') f(\vec{r}') d\vec{r}' - \int_S \varphi(\vec{r}') \frac{\partial G(\vec{r}, \vec{r}')}{\partial n} d\vec{r}'$$

思考 对有界区域 D 上的 Poisson 方程第二边值问题

$$\nabla^2 u(\mathbf{r}) = -f(\mathbf{r}), \quad \mathbf{r} \in D,$$

$$\left. \frac{\partial u(\mathbf{r})}{\partial n} \right|_S = \psi(\mathbf{r}).$$

仿照式 (21) 定义 Green 函数的定解问题为

$$\nabla^2 G(\mathbf{r}, \mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0), \quad \mathbf{r}, \mathbf{r}_0 \in D,$$

$$\left. \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} \right|_{\mathbf{r} \in S} = 0.$$

然后由式 (20) 可得

$$u(\mathbf{r}_0) = \int_D G(\mathbf{r}, \mathbf{r}_0) f(\mathbf{r}) d\mathbf{r} + \int_S G(\mathbf{r}, \mathbf{r}_0) \psi(\mathbf{r}) d\sigma.$$

这样的做法是否有问题? 分别从数学和物理上考虑这一问题.

§13.24 Laplace 方程第一边值问题

$$\begin{cases} \nabla^2 u(\vec{r}) = 0, & \vec{r} \in D \\ u(\vec{r})|_S = \varphi(\vec{r}) \end{cases}$$

$$f(\vec{r}) = 0$$

$$u(\vec{r}_0) = - \int_S \varphi(\vec{r}) \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n} d\sigma$$

$$(\nabla^2 - m^2) G = -\delta(\vec{r})$$

$$G \sim \frac{e^{-mr}}{r}$$

Yukawa 势

§13.3 特殊区域上 Green 函数的制作

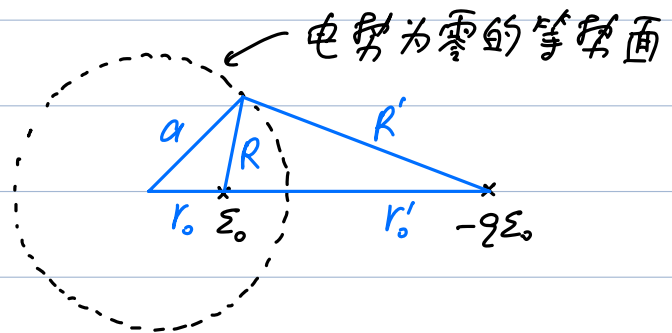
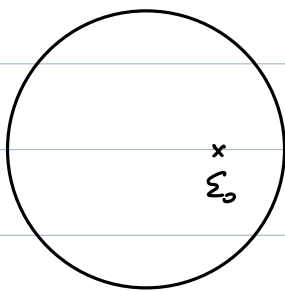
求解 Green 函数的定边问题

镜像法

§13.3.1 球内 Laplace 方程第一边值问题

$$\begin{cases} \nabla^2 u(\vec{r}) = 0, & r < a \\ u(\vec{r})|_{r=a} = f(\theta, \varphi) \end{cases}$$

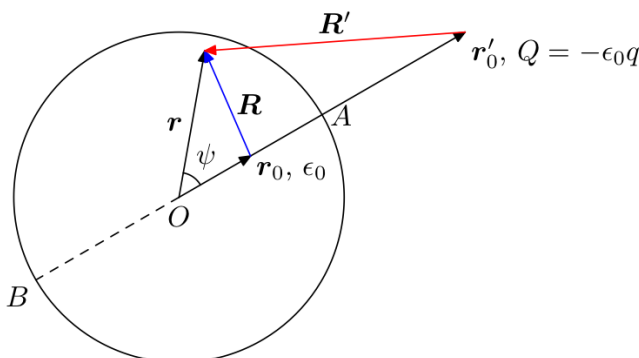
$$\begin{cases} \nabla^2 G(\vec{r}, \vec{r}_0) = -\delta(\vec{r} - \vec{r}_0), & r, r_0 < a \\ G(\vec{r}, \vec{r}_0)|_{r=a} = 0 \end{cases}$$



Appolonius circle

$$\frac{a}{r'} = \frac{r_0}{a} = \frac{R}{R'}$$

$$\frac{1}{R} = \frac{2}{R'}$$



$$G(\vec{r}, \vec{r}_0) = \frac{1}{4\pi R} - \frac{2}{4\pi R'}$$

图 2: 球内区域的镜像法

$$g(\mathbf{r}, \mathbf{r}_0) = \frac{1}{4\pi R} - \frac{q}{4\pi R'} = \frac{1}{4\pi \sqrt{r^2 + r_0^2 - 2rr_0 \cos \psi}} - \frac{q}{4\pi \sqrt{r^2 + r_0'^2 - 2rr_0' \cos \psi}},$$

当然它还依赖于 r_0' 和 q 两个参数. 在球面上, $r = a$, 则

$$g(\mathbf{r}, \mathbf{r}_0)|_{r=a} = \frac{1}{4\pi r_0 \sqrt{1 + (a/r_0)^2 - 2(a/r_0) \cos \psi}} - \frac{q}{4\pi a \sqrt{1 + (r_0'/a)^2 - 2(r_0'/a) \cos \psi}}.$$

只要取 $r_0' = \frac{a^2}{r_0}$, $q = \frac{q}{r_0}$, 则 $g(\mathbf{r}, \mathbf{r}_0)|_{r=a} = 0$

所求的 Green 函数为

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{4\pi R} - \frac{a}{r_0} \frac{1}{4\pi R'},$$

$$R = \sqrt{r^2 + r_0^2 - 2rr_0 \cos \psi}, \quad R' = \sqrt{r^2 + a^4/r_0^2 - 2r(a^2/r_0) \cos \psi},$$

$$\cos \psi = \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos(\phi - \phi_0).$$

有了 Green 函数, 就可以写出定解问题 (34) 的解. 注意球面的外法向方向就是半径方向, 经过一些运算, 可以得到

$$\left. \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} \right|_{r=a} = \left. \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial r} \right|_{r=a} = \left. \frac{r_0^2 - a^2}{4\pi a R^3} \right|_{r=a} = \frac{r_0^2 - a^2}{4\pi a (a^2 + r_0^2 - 2ar_0 \cos \psi)^{3/2}}. \quad (40)$$

代入式 (28), 注意到 $\varphi(\mathbf{r}) = f(\theta, \phi)$ 和 $d\sigma = a^2 \sin \theta d\theta d\phi$, 就得到

$$u(r_0, \theta_0, \phi_0) = \frac{a(a^2 - r_0^2)}{4\pi} \int \frac{f(\theta, \phi) \sin \theta d\theta d\phi}{(a^2 + r_0^2 - 2ar_0 \cos \psi)^{3/2}}. \quad (41)$$

其中 θ 的积分范围是 0 到 π , ϕ 的积分范围是 0 到 2π . 上式称为球的 Poisson 积分. 可以证明, 它与用分离变量法得到的结果是一致的.

§ 13.3.2 上半平面 Laplace 方程第一边值问题

$$\begin{cases} \nabla^2 u(\vec{r}) = 0, & -\infty < x < +\infty, 0 < y < +\infty \\ u(\vec{r})|_{y=0} = \varphi(x) \end{cases}$$

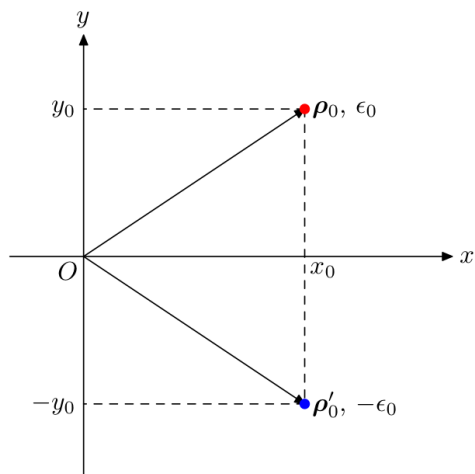


图 3: 上半平面的镜像法

$$\begin{array}{ccc} (x_0, y_0) & & (x_0, y_0) \\ x & & x \\ \hline & \Leftrightarrow & \text{---} \\ & & x \\ & & (x_0, -y_0) \end{array}$$

$$G(\vec{r}, \vec{r}_0) = -\frac{1}{2\pi} \ln |\vec{r} - \vec{r}_0| + \frac{1}{2\pi} \ln |\vec{r} - \vec{r}'_0| = \frac{1}{4\pi} \ln \frac{(x-x_0)^2 + (y+y_0)^2}{(x-x_0)^2 + (y-y_0)^2}$$

$$\frac{\partial G}{\partial n} \Big|_{y=0} = -\frac{\partial G}{\partial y} \Big|_{y=0} = -\frac{y_0}{\pi} \frac{1}{(x-x_0)^2 + y_0^2}$$

$$u(x, x_0) = \frac{y_0}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(x)}{(x-x_0)^2 + y_0^2} dx$$

$$y_0 \rightarrow 0^+, \quad \frac{y_0}{\pi[(x-x_0)^2 + y_0^2]} \rightarrow \delta(x_0 - x)$$

③

§ 13.3.3 圆内 Laplace 方程第一边值问题

$$\begin{cases} \nabla^2 u(\vec{r}) = 0, & r < a \\ u(\vec{r})|_{r=a} = f(\phi) \end{cases}$$

$$\begin{cases} \nabla^2 G(\vec{r}, \vec{r}_0) = -\delta(\vec{r} - \vec{r}_0), & r, r_0 < a \\ G(\vec{r}, \vec{r}_0)|_{r=a} = 0 \end{cases}$$

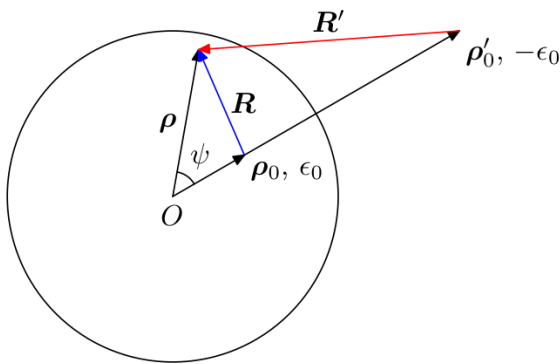


图 4: 圆内区域的镜像法

$$g(\boldsymbol{\rho}, \boldsymbol{\rho}_0) = -\frac{1}{2\pi} \ln R + \frac{1}{2\pi} \ln R' + c = \frac{1}{4\pi} \ln \frac{\rho^2 + a^4/\rho_0^2 - 2\rho(a^2/\rho_0) \cos \psi}{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos \psi} + c. \quad (49)$$

注意其中有一个常数项，它依赖于电势零点的选择，这是二维问题的特点。在圆周上， $\rho = a$ ，易得

$$g(\boldsymbol{\rho}, \boldsymbol{\rho}_0)|_{\rho=a} = \frac{1}{2\pi} \ln \frac{a}{\rho_0} + c \quad (50)$$

由上式可见，只要取

$$c = -\frac{1}{2\pi} \ln \frac{a}{\rho_0}, \quad (51)$$

则 $g(\boldsymbol{\rho}, \boldsymbol{\rho}_0)|_{\rho=a} = 0$ 。将上式所得 c 值代入 $g(\boldsymbol{\rho}, \boldsymbol{\rho}_0)$ ，就得到所求的 Green 函数为

$$G(\boldsymbol{\rho}, \boldsymbol{\rho}_0) = \frac{1}{2\pi} \ln \frac{1}{R} - \frac{1}{2\pi} \ln \left(\frac{a}{\rho_0} \frac{1}{R'} \right) = \frac{1}{4\pi} \ln \frac{\rho_0^2 \rho^2 / a^2 + a^2 - 2\rho\rho_0 \cos \psi}{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos \psi}. \quad (52)$$

有了 Green 函数，就可以写出定解问题 (47) 的解。注意圆周的外法向方向就是半径方向，经过一些运算，可以得到

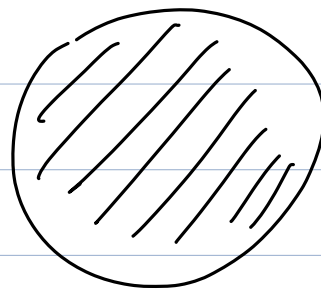
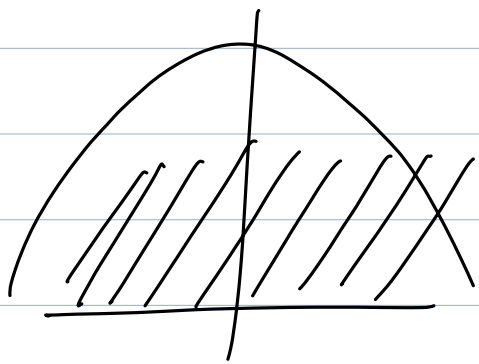
$$\left. \frac{\partial G(\boldsymbol{\rho}, \boldsymbol{\rho}_0)}{\partial n} \right|_{\rho=a} = \left. \frac{\partial G(\boldsymbol{\rho}, \boldsymbol{\rho}_0)}{\partial \rho} \right|_{\rho=a} = \frac{\rho_0^2 - a^2}{2\pi a [a^2 + \rho_0^2 - 2a\rho_0 \cos(\phi - \phi_0)]}. \quad (53)$$

代入式 (28)，注意到 $\varphi(\mathbf{r})$ 现在是 $f(\phi)$ ，而 $d\sigma = a d\phi$ ，就得到

$$u(\rho_0, \phi_0) = \frac{a^2 - \rho_0^2}{2\pi} \int_0^{2\pi} \frac{f(\phi) d\phi}{a^2 + \rho_0^2 - 2a\rho_0 \cos(\phi - \phi_0)}. \quad (54)$$

上式称为圆的 Poisson 积分。可以证明，它与用分离变量法得到的结果是一致的。

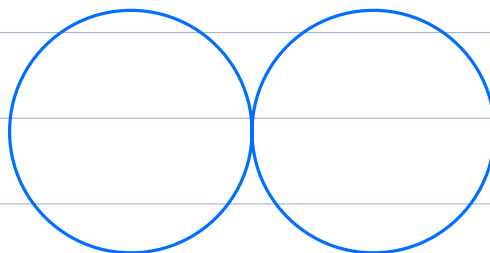
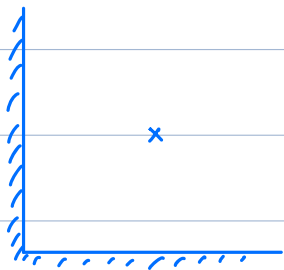
	球/圆内	上半平面
2D	例 3	例 2
3D	例 1	习题



1. 复变函数. Cauchy 积分公式. 取实部. 虚部 Poisson 公式. 2D
2. 分离变量法
3. 格林函数法

Green 函数法

1. 写出格林函数满足的定解问题
2. 求解格林函数
3. 将原来定解问题的解表达成一个积分.



Chap. 12 柱函数初步

Laplace 方程在柱坐标系下分离变量 (回顾)

$$\nabla^2 u = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}$$

$$u = R(\rho) \Phi(\varphi) Z(z)$$

$$\frac{\rho}{R} \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) + \rho^2 \frac{Z''}{Z} = - \frac{\Phi''}{\Phi} = \mu$$

$$\begin{cases} \frac{1}{R\rho} \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) - \frac{\mu}{\rho^2} = - \frac{Z''}{Z} = -\lambda & (\text{侧面加 B.C.}) \\ \Phi'' + \mu \Phi = 0 \end{cases}$$

$$\begin{cases} Z'' - \lambda Z = 0 \end{cases}$$

$$\begin{cases} \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) + \left(\lambda - \frac{m^2}{\rho^2} \right) R = 0 \end{cases}$$

$$x = \sqrt{\lambda} \rho, \quad R(\rho) = y(x) \Rightarrow \text{Bessel eq.}$$

$$y'' + \frac{1}{x} y' + \left(\lambda - \frac{m^2}{x^2} \right) = 0, \quad m \in \mathbb{Z}$$

S-L 形式:

$$\begin{cases} \frac{d}{dx} \left(x \frac{dy}{dx} \right) - \frac{m^2}{x} y + \lambda x y = 0 & (0 < x < a) \end{cases}$$

$$\begin{cases} y(0) = 0 \quad (m=1, 2, \dots) \text{ 或 } |y(0)| < \infty \quad (m=0), \quad \alpha y'(a) + \beta y(a) = 0 \end{cases}$$

↓
自然边界条件

Bessel 方程:

$$y'' + \frac{1}{x} y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0$$

$$\operatorname{Re} \nu > 0$$

我们已经在第十章用级数解法求解了 Bessel 方程，其线性独立解是

$$y(x) = \{J_\nu(x), N_\nu(x)\}. \quad (2)$$

其中

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \nu + 1)} \left(\frac{x}{2}\right)^{2k + \nu}, \quad N_\nu(x) = \frac{J_\nu(x) \cos \nu\pi - J_{-\nu}(x)}{\sin \nu\pi}. \quad (3)$$

是 ν 阶 Bessel 函数和 ν 阶 Neumann 函数. 有些书上将 Neumann 函数记作 $Y_\nu(x)$. 当 $\nu = m \in \mathbb{N}$, $N_\nu(x)$ 为 $0/0$ 型, 可以通过取极限并应用 L' Hospital 法则确定其形式, 这与用级数解法直接求第二解 (并适当选取任意常数) 所得的 $N_m(x)$ 相同, 即 $\lim_{\nu \rightarrow m} N_\nu(x) = N_m(x)$.

Bessel 函数也称为第一类柱函数, Neumann 函数也称为第二类柱函数, 下列 Hankel 函数也称为第三类柱函数:

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x), \quad H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x). \quad (4)$$

$H_\nu^{(1)}(x)$ 和 $H_\nu^{(2)}(x)$ 也分别称为第一种和第二种 Hankel 函数. Bessel 方程的线性独立解可以取为 $\{J_\nu(x), N_\nu(x), H_\nu^{(1)}(x), H_\nu^{(2)}(x)\}$ 中的任意两个.

§12.1 柱函数的基本性质

§12.1.1 原点附近的行为

奇点附近的渐近行为

$$J_\nu(x) \sim x^\nu$$

$$N_\nu(x) \sim C_1 x^\nu + C_2 x^{-\nu}$$

$$x \rightarrow 0 \quad J_\nu \rightarrow 0, \quad J_{-\nu} \rightarrow \infty, \quad N_\nu \rightarrow -\infty \quad (\operatorname{Re} \nu > 0)$$

$$J_0 \sim 1 + \left(\frac{x}{2}\right)^2 \rightarrow 1, \quad N_0 \sim \frac{2}{\pi} \left(\ln \frac{x}{2} + \gamma\right) \rightarrow \infty$$

Series[BesselJ[ν , x], { x , 0, 3}]

$$x^\nu \left(\frac{2^{-\nu}}{\Gamma[1 + \nu]} - \frac{2^{-2-\nu} x^2}{(1 + \nu) \Gamma[1 + \nu]} + O[x]^4 \right)$$

Series[BesselY[ν , x], { x , 0, 3}]

$$x^\nu \left(-\frac{2^{-\nu} \cos[\pi \nu] \Gamma[-\nu]}{\pi} + \frac{2^{-2-\nu} \cos[\pi \nu] \Gamma[-\nu] x^2}{\pi (1 + \nu)} + O[x]^4 \right) + x^{-\nu} \left(-\frac{2^\nu \Gamma[\nu]}{\pi} - \frac{2^{-2+\nu} \Gamma[\nu] x^2}{\pi (-1 + \nu)} + O[x]^4 \right)$$

§12.1.2 递推关系

$$Z_\nu(x) \quad \{J_\nu(x), N_\nu(x), H_\nu^{(1)}(x), H_\nu^{(2)}(x)\}$$

$$\frac{d}{dx} [x^\nu Z_\nu(x)] = x^\nu Z_{\nu-1}(x), \quad \frac{d}{dx} [x^{-\nu} Z_\nu(x)] = -x^{-\nu} Z_\nu(x)$$

$$Z = J \quad J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

$$x^\nu J_\nu = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+\nu+1)} \frac{x^{2k+2\nu}}{2^{2k+\nu}}$$

$$\frac{d}{dx} (x^\nu J_\nu) = \sum_{k=0}^{\infty} (2k+2\nu) \frac{(-1)^k}{k! \Gamma(k+\nu+1)} \frac{x^{2k+2\nu-1}}{2^{2k+\nu}}$$

$$x^\nu J_{\nu-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+\nu)} \frac{x^{2k+2\nu-1}}{2^{2k+\nu-1}}$$

$$\nu=0: \quad J_0'(x) = -J_1(x)$$

$$\int J_1(x) dx = -J_0(x) + C$$

$$Z_{\nu-1}(x) + Z_{\nu+1}(x) = \frac{2\nu}{x} Z_\nu(x), \quad Z_{\nu-1}(x) - Z_{\nu+1}(x) = 2Z_\nu'(x)$$

柱函数

§12.1.3 渐近形式

$$x = \infty$$

非正则奇点

$$H_\nu^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{i(x-\nu\pi/2-\pi/4)},$$

$$e^{ix}$$

$$H_\nu^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{-i(x-\nu\pi/2-\pi/4)},$$

$$e^{-ix}$$

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right),$$

$$\cos x$$

$$N_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

$$\sin x$$

Series[HankelH1[v, x], {x, Infinity, 3}, Assumptions -> x > 0]

$$e^{i x - \frac{1}{4} i \pi (-3+2 v) + 0 \left[\frac{1}{x} \right]^4} \left(-\sqrt{\frac{2}{\pi}} \sqrt{\frac{1}{x}} - \frac{i (-1+4 v^2) \left(\frac{1}{x} \right)^{3/2}}{4 \sqrt{2} \pi} + \frac{(9-40 v^2+16 v^4) \left(\frac{1}{x} \right)^{5/2}}{64 \sqrt{2} \pi} + 0 \left[\frac{1}{x} \right]^{7/2} \right)$$

Series[HankelH2[v, x], {x, Infinity, 3}, Assumptions -> x > 0]

$$e^{-i x + \frac{1}{4} i \pi (3+2 v) + 0 \left[\frac{1}{x} \right]^4} \left(-i \sqrt{\frac{2}{\pi}} \sqrt{\frac{1}{x}} - \frac{(-1+4 v^2) \left(\frac{1}{x} \right)^{3/2}}{4 \sqrt{2} \pi} + \frac{i (9-40 v^2+16 v^4) \left(\frac{1}{x} \right)^{5/2}}{64 \sqrt{2} \pi} + 0 \left[\frac{1}{x} \right]^{7/2} \right)$$

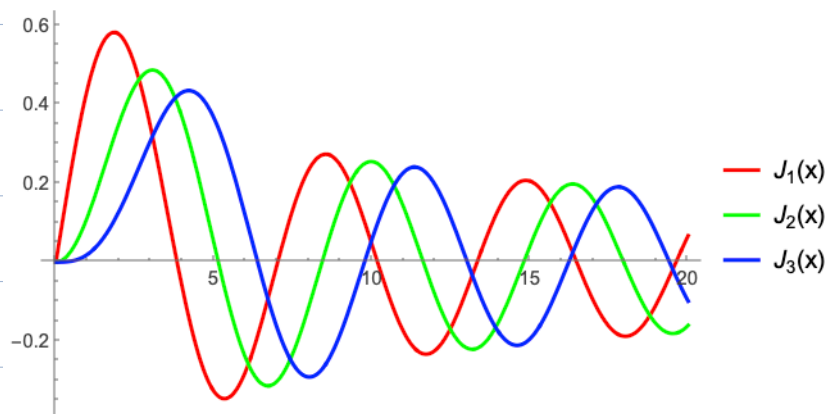
Series[BesselJ[v, x], {x, Infinity, 2}]

$$\cos \left[x - \frac{1}{4} \pi (1+2 v) + 0 \left[\frac{1}{x} \right]^3 \right] \left(\sqrt{\frac{2}{\pi}} \sqrt{\frac{1}{x}} + 0 \left[\frac{1}{x} \right]^{5/2} \right) + \left(\frac{(1-4 v^2) \left(\frac{1}{x} \right)^{3/2}}{4 \sqrt{2} \pi} + 0 \left[\frac{1}{x} \right]^{5/2} \right) \sin \left[x - \frac{1}{4} \pi (1+2 v) + 0 \left[\frac{1}{x} \right]^3 \right]$$

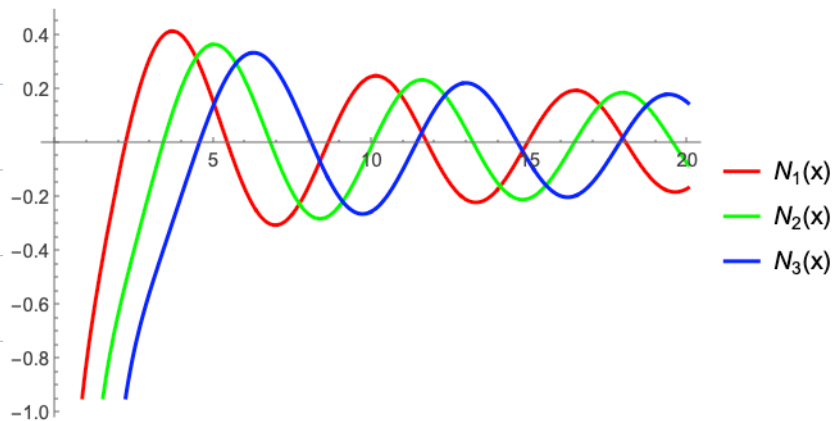
Series[BesselY[v, x], {x, Infinity, 2}]

$$\cos \left[x - \frac{1}{4} \pi (1+2 v) + 0 \left[\frac{1}{x} \right]^3 \right] \left(\frac{(-1+4 v^2) \left(\frac{1}{x} \right)^{3/2}}{4 \sqrt{2} \pi} + 0 \left[\frac{1}{x} \right]^{5/2} \right) + \left(\sqrt{\frac{2}{\pi}} \sqrt{\frac{1}{x}} + 0 \left[\frac{1}{x} \right]^{5/2} \right) \sin \left[x - \frac{1}{4} \pi (1+2 v) + 0 \left[\frac{1}{x} \right]^3 \right]$$

Plot[{BesselJ[1, x], BesselJ[2, x], BesselJ[3, x]}, {x, 0, 20}, PlotStyle -> {Red, Green, Blue}, PlotLegends -> {"J₁(x)", "J₂(x)", "J₃(x)"}]



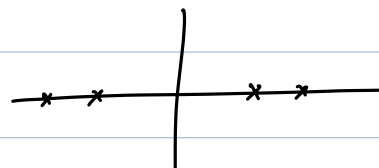
Plot[{BesselY[1, x], BesselY[2, x], BesselY[3, x]}, {x, 0, 20}, PlotStyle -> {Red, Green, Blue}, PlotLegends -> {"N₁(x)", "N₂(x)", "N₃(x)"}]



§12.1.4 零点

柱函数 $J_\nu(x)$ 的任何零点 x_0 都是简单零点, 只有 $x_0=0$ 可能例外.

$$J_\nu(e^{i\pi}x) = e^{i\nu\pi} J_\nu(x)$$



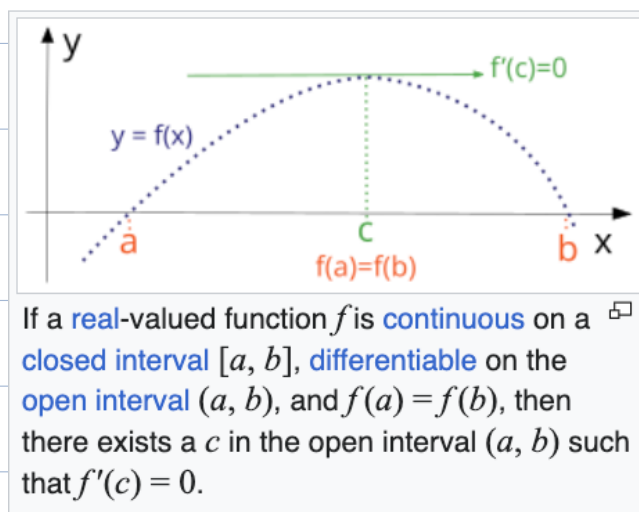
(1) $J_\nu(x)$ 有无穷多正零点

$J'_\nu(x)$ — — — — —

第一类 BC

=

Rolle 定理



(2) $J_\nu(x)$ 和 $J_{\nu+1}(x)$ 的正零点是两两相间的. 如果 $\nu > -1$, 则 $J_\nu(x)$ 的最小正零点更靠近原点.

$$\frac{d}{dx} [x^{\nu+1} J_{\nu+1}(x)] = x^{\nu+1} J_\nu(x)$$

(3) $AJ_\nu(x) + BxJ'_\nu(x)$ 有无穷多个正零点, 其中 $A, B \in \mathbb{R}$ 且不全为零.

(4) $N_\nu(x)$ 有无穷多正零点, 而且与 $J_\nu(x)$ 的正零点两两相间.

(5) 当 $\nu > -1$, $J_\nu(x)$ 的零点都是实数

§12.2 整数阶柱函数

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k+m}$$

$$J_{-m}(x) = (-1)^m J_m(x), \quad N_{-m}(x) = (-1)^m N_m(x)$$

§12.2.1 对称性

$$J_m(-x) = (-1)^m J_m(x)$$

$J_m(x)$ 的奇偶性与 m 一致

§12.2.2 零点

(1) $J_m(x)$ 有无穷多实数零点, 且正负成对. $J'_m(x)$ 亦然.

注意 $J_m(x)$ 是单值函数, 所以负实轴上与 $x > 0$ 对称的点可以直接写成 $-x$, 而不必写成 $e^{i\pi}x$.

$J_m(x)$ 的正零点从小到大排列, 记作 $\{x_{mn}\}_{n=1}^{\infty}$, 满足

$$\sin kx = 0$$

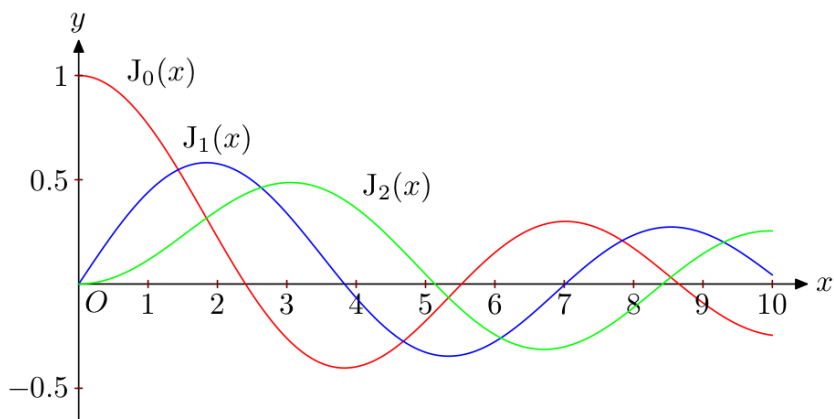
$$0 < x_{m1} < x_{m2} < \cdots \quad (16)$$

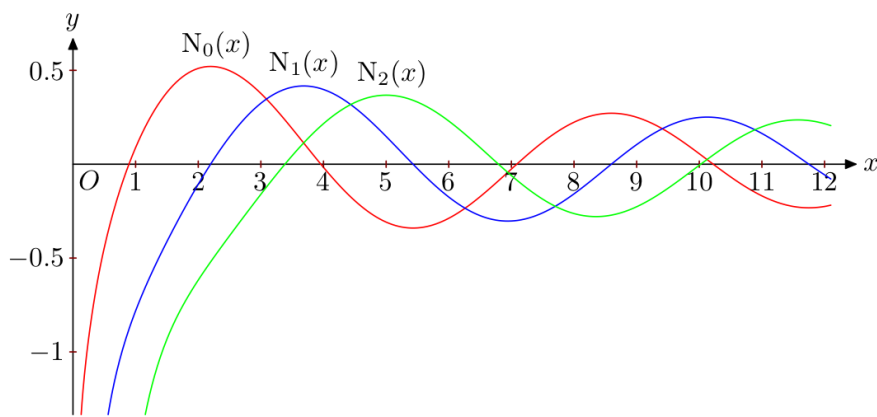
$J'_m(x)$ 的正零点从小到大排列, 记作 $\{\tilde{x}_{mn}\}_{n=1}^{\infty}$, 满足

$$0 < \tilde{x}_{m1} < \tilde{x}_{m2} < \cdots \quad (17)$$

(2) $J_m(x)$ 和 $J_{m+1}(x)$ 的正零点是两两相间的. $J_m(x)$ 的最小正零点比 $J_{m+1}(x)$ 的更靠近原点, 即 $x_{m1} < x_{m+1,1}$.

§12.2.3 函数图像





§12.2.4 母函数

$$e^{\frac{z}{2}(t - \frac{1}{t})} = \sum_{m=-\infty}^{\infty} J_m(z) t^m, \quad 0 < |t| < +\infty$$

$$\frac{1}{2} \sum_{m=-\infty}^{\infty} [J_{m-1}(z) - J_{m+1}(z)] t^m = \sum_{m=-\infty}^{\infty} J'_m(z) t^m$$

§12.2.5 加法公式

$$J_m(x+y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{m-k}(y)$$

§12.2.6 积分表示

$$t = e^{i\phi}$$

$$e^{ix \sin \phi} = \sum_{m=-\infty}^{\infty} J_m(x) e^{im\phi}$$

$$J_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix \sin \phi} e^{-im\phi} d\phi = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin \phi - m\phi) d\phi$$

$$|J_m(x)| \leq 1, \quad x \in \mathbb{R}$$

§12.2.7 含 Bessel 函数的积分

$$I = \int_0^{\infty} e^{-ax} J_0(bx) dx, \quad a, b > 0$$

$$= \int_0^{\infty} e^{-ax} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix \sin \phi} d\phi \right) dx = \dots = \frac{1}{\sqrt{a^2 + b^2}}$$

§ 12.2.8 本征值问题

对 Laplace 方程或 Helmholtz 方程在柱坐标系中分离变量，会遇到下述方程

$$y'' + \frac{1}{x}y' + \left(\lambda - \frac{m^2}{x^2} \right) y = 0, \quad m \in \mathbb{N}. \quad (31)$$

将它改写成 Sturm-Liouville 形式，结合边界条件，典型的本征值问题是：

$$\frac{d}{dx} \left(x \frac{dy}{dx} \right) - \frac{m^2}{x} y + \lambda xy = 0, \quad (0 < x < a) \quad (32a)$$

$$y(0) = 0 \ (m = 1, 2, \dots) \quad \text{或} \quad |y(0)| < \infty \ (m = 0), \quad \alpha y'(a) + \beta y(a) = 0, \quad (32b)$$

其中 $\alpha, \beta \geq 0$ ，但 α 和 β 不同时为 0； $x = 0$ 处的边界条件是自然边界条件。与 Sturm-Liouville 方程比较可知 $k(x) = x \geq 0$ ， $q(x) = m^2/x \geq 0$ ， $\rho(x) = x \geq 0$ 。根据 Sturm-Liouville 本征值问题的一般结论可知， $\lambda \geq 0$ ，故可令 $\lambda = k^2$ ，其中 $k \geq 0$ 。代入方程 (31)，其解为

$$y(x) = \{J_m(kx), N_m(kx)\}. \quad (33)$$

考虑到 $x = 0$ 处的自然边界条件， $N_m(kx)$ 显然不符合要求，应该舍弃。故

$$y(x) = J_m(kx). \quad (34)$$

由 $x = a$ 处的边界条件，可得确定本征值的超越方程为

$$\alpha k J'_m(ka) + \beta J_m(ka) = 0. \quad (35)$$

$$0 < k_1 < k_2 < \dots$$

$$\lambda_n = k_n^2, \quad y_n(x) = J_m(k_n x), \quad n = 1, 2, \dots$$

$$y'' + \frac{1}{x} y' + \left(\lambda - \frac{m^2}{x^2}\right) y = 0$$

$$z'' + \lambda z = 0$$

$$y = C_1 J_m(kx) + C_2 N_m(kx)$$

$$z = C_1 \sin \sqrt{\lambda} x + C_2 \cos \sqrt{\lambda} x$$

$$|y(0)| < \infty, \quad y(a) = 0$$

$$\hookrightarrow y = C_1 J_m(kx)$$

$$z(0) = 0, \quad z(l) = 0$$

$$\hookrightarrow z = C_1 \sin \sqrt{\lambda} x$$

$$y(a) = 0 \Rightarrow J_m(ka) = 0$$

$$\Rightarrow \lambda_n = k_n^2, \quad n = 1, 2, \dots$$

$$z(l) = 0 \Rightarrow \sin \sqrt{\lambda} l = 0$$

$$\Rightarrow \sqrt{\lambda}_n = \frac{n\pi}{l}, \quad n = 1, 2, \dots$$

本征函数 $y_n(x) = J_m(k_n x)$

本征函数 $z_n = \sin \frac{n\pi x}{l}$

$$(y_i(x), y_j(x)) = N_i^2 \delta_{ij}$$

$$(z_n, z_m) = \frac{l}{2} \delta_{nm}$$

$$f(x) = \sum_{n=0}^{\infty} f_n J_m(k_n x), \quad 0 < x < a$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{l}$$

$$f_n = \frac{1}{\|J_m(k_n x)\|^2} \int_0^a f(x) J_m(k_n x) x dx$$

$$f_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$\int_0^a J_m(k_i x) J_m(k_j x) x dx = \|J_m(k_n x)\|^2 \delta_{ij}$$

§12.2.9 本征函数的正交关系

§12.2.10 本征函数的模

§12.2.11 广义 Fourier 级数

§12.2.12 应用

例 1 无穷长均匀导热圆柱体, 半径为 ρ_0 , 侧面保持温度为零度, 柱内的初始温度分布为已知函数 $f(\rho, \phi)$, 求柱内各处的温度变化.

采用柱坐标系, 以圆柱的对称轴为 z 轴, 任取轴上一点为原点. 柱内的温度分布 $u(\mathbf{r}, t)$ 满足定解问题

$$\frac{\partial u}{\partial t} - a^2 \nabla^2 u = 0, \quad \rho < \rho_0, t > 0, \quad (48a)$$

$$u|_{\rho=\rho_0} = 0, \quad (48b)$$

$$u|_{t=0} = f(\rho, \phi). \quad (48c)$$

分离变量, 令 $u(\mathbf{r}, t) = v(\mathbf{r})T(t)$, 代入上式中的方程和边界条件, 可得常微分方程

$$T' + \lambda a^2 T = 0 \quad (49)$$

和本征值问题

$$\nabla^2 v + \lambda v = 0, \quad \rho < \rho_0, \quad (50a)$$

$$v|_{\rho=\rho_0} = 0. \quad (50b)$$

由于圆柱无穷长, 初始条件和边界条件均与 z 无关, 所以 u 和 v 也与 z 无关. 进一步分离变量, 令 $v(\mathbf{r}) = R(\rho)\Phi(\phi)$, 代入式 (50), 考虑到 $\Phi(\phi)$ 满足周期性边界条件, 可得

$$\Phi(\phi) = \{\cos m\phi, \sin m\phi\}, \quad m = 0, 1, 2, \dots$$

将求解 $\Phi(\phi)$ 所得的本征值代入 $R(\rho)$ 的方程, 并考虑到边界条件, 可知 $R(\rho)$ 满足本征值问题

$$R'' + \frac{1}{\rho}R' + \left(\lambda - \frac{m^2}{\rho^2}\right)R = 0, \quad 0 < \rho < \rho_0,$$

$$R(0) = 0 \quad (m \neq 0) \quad \text{或} \quad |R(0)| < \infty \quad (m = 0); \quad R(\rho_0) = 0.$$

其中 $\rho = 0$ 处的边界条件是自然边界条件, 第九章曾经详细讨论过. 这已经是前面求解过的本征值问题, 现在是第一类齐次边界条件, 所以得到下列结果. 对于给定的 m , 本征值为

$$\lambda_{mn} = k_{mn}^2, \quad k_{mn} = \frac{x_{mn}}{\rho_0}, \quad n = 1, 2, \dots \quad (51)$$

相应的本征函数为

$$R_{mn}(\rho) = J_m(k_{mn}\rho), \quad n = 1, 2, \dots$$

将本征值代回式 (49), 易得 $T_{mn}(t) = \exp(-\lambda_{mn}a^2t)$. 将各本征解叠加, 得到一般解为

$$u(\rho, \phi, t) = \sum_{n=1}^{\infty} A_{0n} J_0(k_{0n}\rho) \exp(-\lambda_{0n}a^2t) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) (A_{mn} \cos m\phi + B_{mn} \sin m\phi) \exp(-\lambda_{mn}a^2t). \quad (52)$$

§12.3 虚宗量 Bessel 函数

$$y'' + \frac{1}{x} y' - \left(1 + \frac{\nu^2}{x^2}\right) y = 0$$

$$y'' + \frac{1}{x} y' + \left(i^2 - \frac{\nu^2}{x^2}\right) y = 0$$

$$\text{令 } t = ix$$

$$y = Z_\nu(ix)$$



§12.3.1 虚宗量 Bessel 函数

变型 - - - - - (Modified Bessel function)

$$I_\nu(x) = e^{-i\nu\pi/2} J_\nu(e^{i\pi/2} x)$$

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

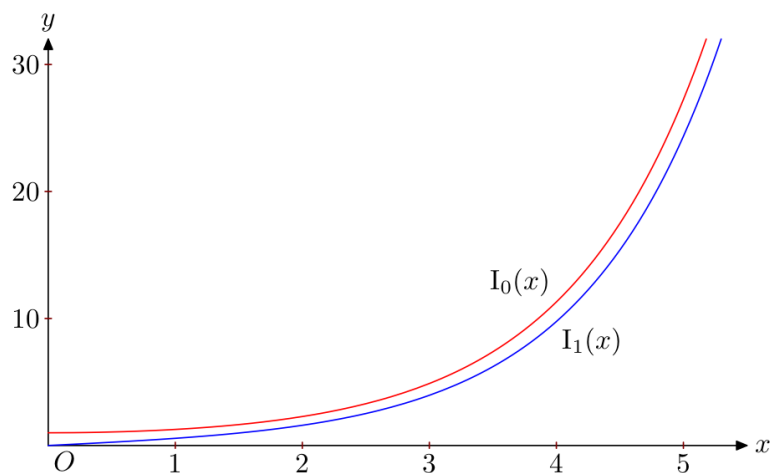


图 3: 虚宗量 Bessel 函数的图像

$$I_0(0) = 1, \quad I_m(0) = 0, \quad m \in \mathbb{N}^+$$

$$\frac{d}{dx} [x^\nu I_\nu(x)] = x^\nu I_{\nu-1}(x), \quad \frac{d}{dx} [x^{-\nu} I_\nu(x)] = x^{-\nu} I_{\nu+1}(x)$$

§12.3.2 虚宗量 Hankel 函数

第二类虚宗量/变型 Bessel 函数

Bessel I, Bessel K

$$K_\nu(x) = \frac{\pi}{2\sin\nu\pi} (I_{-\nu}(x) - I_\nu(x))$$

$$K_\nu(x) = \frac{1}{2} i\pi e^{i\nu\pi/2} H_\nu^{(1)}(e^{i\pi/2}x)$$

假设 $\operatorname{Re} \nu \geq 0$,

$$\nu \notin \mathbb{N}: \quad x \rightarrow 0, \quad I_\nu(x) \rightarrow 0, \quad K_\nu(x) \rightarrow \infty$$

$$\nu = m \in \mathbb{N} \quad x \rightarrow 0, \quad I_0(x) \rightarrow 1, \quad I_m(x) \rightarrow 0 \quad (m \in \mathbb{N}^+), \quad K_m(x) \rightarrow \infty$$

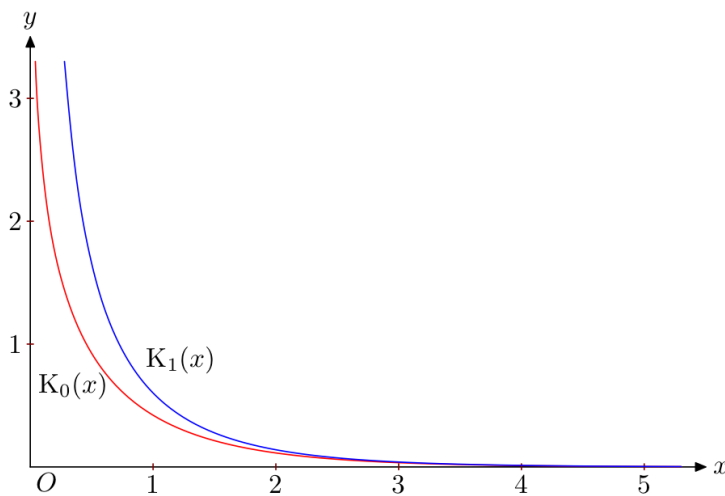


图 4: 虚宗量 Hankel 函数的图像

$x \rightarrow 0$	$I_\nu(x) \checkmark$
$x \rightarrow \infty$	$K_\nu(x) \checkmark$

§12.3.3 渐近形式

$I_\nu(x)$ 和 $K_\nu(x)$ 的渐近形式可以由 $J_\nu(x)$ 和 $H_\nu^{(1)}(x)$ 的渐近形式导出, 结果如下:

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}, \quad x \rightarrow \infty, -\frac{\pi}{2} < \arg x < \frac{\pi}{2}.$$

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x \rightarrow \infty, -\pi < \arg x < \pi.$$

§12.3.4 应用

例 1 均匀导热圆柱体, 半径为 a , 高为 h , 侧面有恒定热流 $q(z)$ 流入, 上下底保持温度 u_0 , u_0 为常数, 求柱内的稳定温度分布.

采用柱坐标, 以圆柱的对称轴为 z 轴, 方向由下底指向上底, 以下底中心为原点. 由于边界条件与 ϕ 无关, 故柱内的稳定温度分布 $u(\mathbf{r})$ 也与 ϕ 无关, 即 $u(\mathbf{r}) = u(\rho, z)$, 它满足定解问题

$$\nabla^2 u = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{\partial^2 u}{\partial z^2} = 0, \quad \rho < a, 0 < z < h, \quad (79a)$$

$$\left. \frac{\partial u}{\partial \rho} \right|_{\rho=a} = \frac{1}{k} q(z), \quad (79b)$$

$$u|_{z=0} = u_0, \quad u|_{z=h} = u_0. \quad (79c)$$

为了将边界条件齐次化, 令 $u(\rho, z) = u_0 + v(\rho, z)$, 则 $v(\rho, z)$ 满足的定解问题为

$$\nabla^2 v = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial v}{\partial \rho} \right) + \frac{\partial^2 v}{\partial z^2} = 0, \quad \rho < a, 0 < z < h, \quad (80a)$$

$$\left. \frac{\partial v}{\partial \rho} \right|_{\rho=a} = \frac{1}{k} q(z), \quad (80b)$$

$$v|_{z=0} = 0, \quad v|_{z=h} = 0. \quad (80c)$$

分离变量, 令 $v(\rho, z) = R(\rho)Z(z)$, 可得方程

$$R'' + \frac{1}{\rho} R' - \lambda R = 0, \quad (81)$$

和本征值问题

$$\begin{aligned} Z'' + \lambda Z &= 0, \\ Z(0) &= 0, \quad Z(h) = 0. \end{aligned}$$

$$\lambda_n = \left(\frac{n\pi}{h} \right)^2, \quad Z_n(z) = \sin \frac{n\pi}{h} z, \quad n = 1, 2, \dots \quad (82)$$

将本征值代回式 (81), 易得其解为

$$R_n(\rho) = \left\{ I_0 \left(\frac{n\pi}{h} \rho \right), K_0 \left(\frac{n\pi}{h} \rho \right) \right\}.$$

但是 $K_0(n\pi\rho/h)$ 在 $\rho = 0$ 处有奇性, 应该舍弃, 故

$$R_n(\rho) = C_n I_0 \left(\frac{n\pi}{h} \rho \right). \quad (83)$$

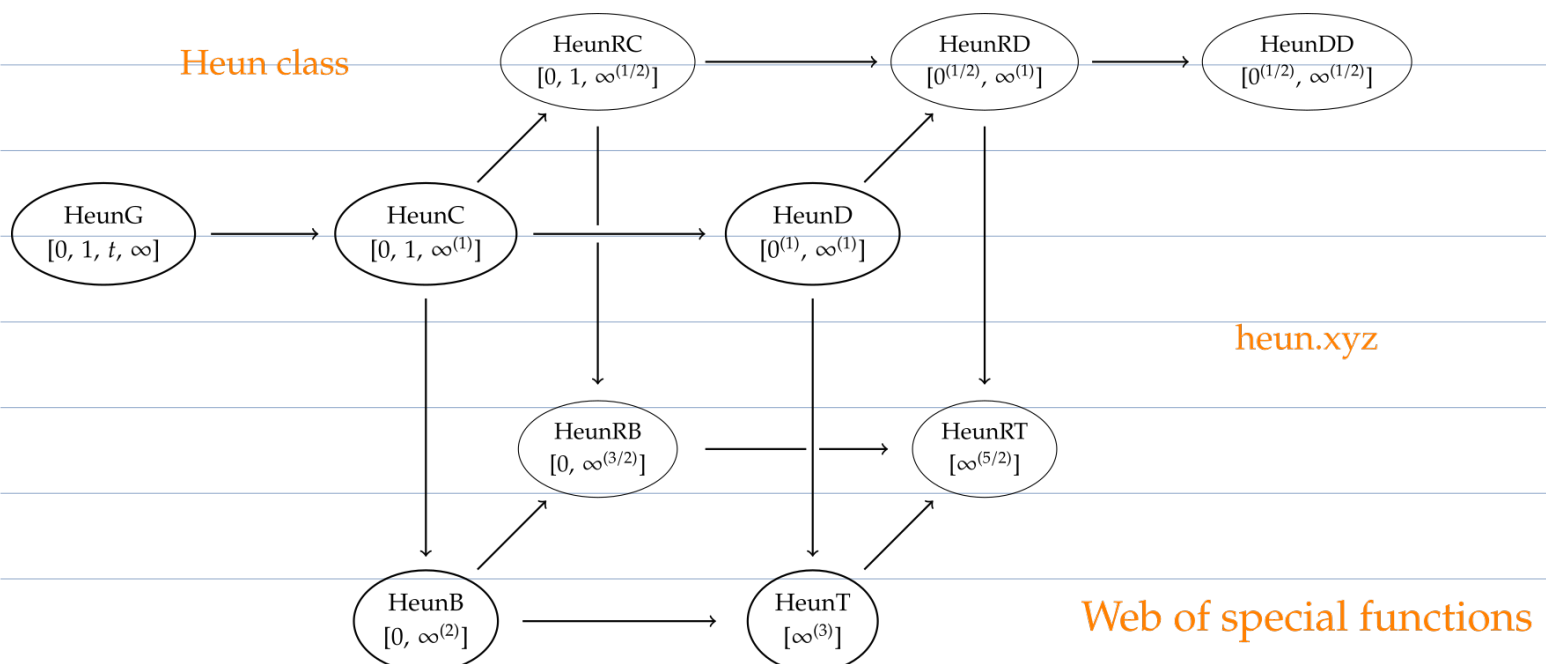
一般解可以取为

$$v(\rho, z) = \sum_{n=1}^{\infty} A_n \frac{I_0(n\pi\rho/h)}{I_1(n\pi a/h)} \sin \frac{n\pi}{h} z. \quad (84)$$



Classifying ODEs by singularities

- ▶ Three regular singularities: Hypergeometric equation
 - ▶ hypergeometric function ${}_2F_1(a, b; c; z)$
 - ▶ Legendre functions $P_n(x)$, $Q_n(x)$, $P_l^m(x)$, $Q_l^m(x)$
 - ▶ Gegenbauer polynomials $C_n^{(m)}(x)$
 - ▶ Chebyshev polynomials $T_n(x)$
- ▶ Confluent hypergeometric equation
 - ▶ confluent hypergeometric function ${}_1F_1(a; b; z)$, $U(a; b; z)$
 - ▶ Whittaker function $M_{k,m}(z)$, $W_{k,m}(z)$
 - ▶ Bessel functions: $J_\nu(z)$, $N_\nu(z)$, $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$, $I_\nu(z)$, $K_\nu(z)$
 - ▶ Laguerre polynomials $L_n(x)$
 - ▶ Parabolic cylinder function $D_\nu(z)$
 - ▶ Hermite polynomials $H_n(x)$
 - ▶ Airy function $\text{Ai}(z)$, $\text{Bi}(z)$
- ▶ Four regular singularities: Heun equation
Heun function $\text{HeunG}(t, q; \alpha, \beta, \gamma, \delta; z)$
- ▶ Confluent cases: HeunC, HeunD, HeunB, HeunT





Wir müssen wissen. Wir werden wissen.

We must know. We shall know. — David Hilbert